INELASTIC NUCLEAR SCATTERING OF STRONGLY ABSORBED PARTICLES

by

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CONTENTS

ABSTRACT

Closed expressions for differential cross sections are derived from the Austern-Blair theory of inelastic nuclear scattering. These formulae are an extension of the strong absorption model of Frahn and Venter for elastic scattering. All the well-known features of inelastic angular distributions of composite particles are shown explicitly. These include effects of the multipolarity of the transition, the nuclear interaction, the Coulomb force and the shape of the reflection coefficients in angular momentum space. Analyses of representative sets of experimental data are presented. The results indicate that the Austern-Blair theory is remarkably successful in representing the shape of the angular distribution, but tends to underestimate the deformation parameter.

I. INTRODUCTION

During the past few years considerable effort has been devoted to the study of collective nuclear states by means of the inelastic scattering of medium energy particles. Collective excitations are amenable to study because they have large transition probabilities. It is, furthermore, possible to describe these states in a relatively simple way in terms of vibrations or rotations of the nuclear surface [1].

The angular distributions of the elastic and inelastic scattering of composite particles bear a close relation to one another. If the Coulomb interaction is weak, both distributions exhibit a clear oscillatory diffraction structure. The oscillations of the inelastic angular distribution are in phase with those of the elastic angular distribution for odd angular momentum transfer, and are out of phase for even angular momentum transfer [2]. In the presence of strong Coulomb interaction, such as for heavy ion scattering, the elastic and inelastic angular distributions are smoothed out to the same extent.

The first explanation for the phase relation mentioned was given in terms of semi-classical models [2-5]. These models employ simple geometrical pictures of the interaction process, which are valid for high bombarding energies [6]. A more satisfactory treatment has been given by Blair, Sharp and Wilets [7] in terms of partial wave amplitudes. Their treatment, however, is restricted to quadrupole excitation and neglects the Coulomb interaction.

If the inelastic transition is weak in comparison with the elastic scattering, the distorted wave Born approximation (DWBA) persion relations of high energy physics".

In order to make the connection between inelastic and elastic scattering more explicit, closed expressions have been derived [13] from the Austern-Blair (AB) theory by means of a consistent approximation method. The main assumption is the strong absorption condition already used in the AB theory. In this way one obtains a generalization of the Fraunhofer diffraction model of Blair [2], while retaining its simplicity. The resulting formulae are an extension of the strong absorption model (SAM) formalism of Frahn and Venter [10] for elastic scattering. They display all the characteristic features of the elastic and inelastic scattering of strongly absorbed particles. If suitable parametrizations [10] are introduced for the reflection coefficients, it becomes possible to describe elastic and inelastic scattering directly in terms of the asymptotic properties of the wave function. The expressions include as special cases the formulae of Bassichis and Dar [14, 15] and Hahne [16].

In section II a brief outline is given of the basic distorted wave theory. In section III the Austern-Blair relations are derived in a somewhat different fashion from that used in ref [12], and the assumptions discussed. The case of mutual excitation of the reaction partners as well as the inelastic scattering of identical particles are treated. In section IV the closed formalism is developed, various special cases are considered, and the physical implications are discussed. Finally, analyses of representative sets of experimental data in terms of the SAM formulae are presented in section V.

II. DISTORTED WAVE THEORY

Since it is at present not possible to solve the nuclear many-body problem, simplified models are used to describe nuclear reactions. For instance, in the case of elastic neutron scattering, the optical potential has been introduced [17] to describe the relative motion of the colliding particles. The excitation of collective states via inelastic scattering can be described in terms of an extended optical potential [18] which depends on the dynamical variables of the nuclear surface.

The deformation of the nuclear surface is defined in terms of a spherical harmonic expansion

$$R = R_0 + \alpha \tag{1}$$

where

$$\alpha = \sum_{LM} \xi_{LM} \, Y_L^{M*}(\Omega_{\vec{r}}) \, . \tag{2}$$

The ξ_{LM} are operators which describe the nuclear surface motion, and R_0 is the equilibrium radius. The extended optical potential, as a function of the distance from the nuclear surface, is

$$U_{ext.} = U(r - R_0) + \Delta U \tag{3}$$

where U describes the elastic scattering and ΔU is the contribution which causes inelastic transitions.

The total Hamiltonian of the system is given by

$$H_T = H(\xi) + T + U + \Delta U \tag{4}$$

where $H(\xi)$ refers to the internal motion of the colliding nuclei, and T is the kinetic energy operator. The transition amplitude for exciting a nucleus in the intrinsic state $\underset{\sim}{a}$ to a final state $\underset{\sim}{b}$ is

$$T_{ba} = \langle b \, \chi^{(-)} | \Delta U | \Psi^{(+)} \rangle \tag{5}$$

where

$$H_T \Psi^{(+)} = E \Psi^{(+)} . \tag{6}$$

The ($\pm$) superscripts refer to outgoing and ingoing wave boundary conditions respectively.

In order to retain all second order terms in T_{ba}, the total wave function $\Psi^{(+)}$ must be approximated to first order [19]

$$\Psi^{(+)} \approx (1 + G \Delta U)\, a\, \chi^{(+)} , \tag{7}$$

where G is the Green's function operator

$$G = \left[E - T - U - H(\xi) + i\epsilon \right]^{-1} . \tag{8}$$

Then, up to second order in ΔU, the DWBA transition amplitude consists of a sum of single and double excitation amplitudes

$$T_{ba}(\mathrm{DWBA}) = T_{ba}(1) + T_{ba}(2) , \tag{9}$$

where

$$T_{ba}(1) = \left\langle b\, \chi^{(-)}(\vec{k}_b, \vec{r}_1) \right| \alpha \frac{\partial U}{\partial R_0} \left| a\, \chi^{(+)}(\vec{k}_a, \vec{r}_2) \right\rangle \tag{10}$$

and

$$T_{ba}(2) = \left\langle b\, \chi^{(-)}(\vec{k}_b, \vec{r}_1) \right| \frac{\alpha^2}{2!} \frac{\partial^2 U}{\partial R_0^2} + \alpha \frac{\partial U}{\partial R_0} G\, \alpha \frac{\partial U}{\partial R_0} \left| a\, \chi^{(+)}(\vec{k}_a, \vec{r}_2) \right\rangle . \tag{11}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega} = (\mu / 2\pi \hbar^2)^2 (k_b / k_a) \sum_{\text{ave.}} \left| T_{ba} \right|^2 \tag{12}$$

where μ is the reduced mass of the system.

The $\chi^{(\pm)}(\vec{k}, \vec{r})$ are the distorted waves for the elastic scattering of a projectile with incident momentum $\hbar \vec{k}_a$ and final momentum $\hbar \vec{k}_b$. They are connected through the relation

$$\chi^{(-)*}(\vec{k}, \vec{r}) = \chi^{(+)}(-\vec{k}, \vec{r}) . \tag{13}$$

These functions are generated from the Schroedinger equation

$$\left[\nabla^2 + k^2 - \frac{2\mu}{\hbar^2}\, U(r) - \frac{2\mu}{\hbar^2}\, U_c(r)\right] \chi(\vec{k},\vec{r}) = 0 \quad , \tag{14}$$

where U_c is the Coulomb potential. For a uniformly charged sphere of radius R_c,

$$\begin{aligned} \frac{2\mu}{\hbar^2} U_c &= \frac{kn}{R_c}\left(3 - \frac{r^2}{R_c^2}\right) \quad , \quad r \le R_c \\ &= 2kn/r \quad , \quad r \ge R_c \ . \end{aligned} \tag{15}$$

The Coulomb parameter n is defined by

$$n = \frac{\mu z Z e^2}{\hbar^2 k} \tag{16}$$

where e is the electron charge and z, Z are the charge numbers of the projectile and target nucleus, respectively.

In practice distorted wave calculations are carried out in terms of partial waves. Partial wave expansion of the scattering wave functions

$$\chi^{(+)} = \frac{4\pi}{kr} \sum_{\ell m} i^\ell e^{i\sigma_\ell} f_\ell(k,r)\, Y_\ell^{m*}(\Omega_{\vec{k}})\, Y_\ell^m(\Omega_{\vec{r}}) \tag{17}$$

defines the radial functions f_ℓ which satisfy the radial Schroedinger equation

$$\left[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - \frac{2\mu}{\hbar^2} U(r) - \frac{2\mu}{\hbar^2} U_c(r)\right] f_\ell(k,r) = 0 \ , \tag{18}$$

with the boundary conditions

$$\begin{aligned} f_\ell\,(k,0) &= 0 \\ f_\ell \xrightarrow[(r\to\infty)]{} &\ \frac{i}{2}\left(e^{-i\theta_\ell} - \eta_\ell\, e^{i\theta_\ell}\right) \end{aligned} \tag{19}$$

where

$$\theta_\ell = kr - n \log 2kr - \ell\pi/2 + \sigma_\ell \ . \tag{20}$$

The Coulomb phase shift is given by

$$\sigma_\ell = \sigma_0(n) + \sum_{s=1}^{\ell} \arctan\left(\frac{n}{s}\right) \ . \tag{21}$$

The reflection coefficients η_ℓ determine the elastic angular distribution.

Consider the scattering of a spin zero particle exciting an even nucleus to a final state of angular momentum L, with projection M. Then the nuclear matrix element $\langle b|\alpha|a\rangle$ contained in eq. (10) can be written [12]:

$$\langle b(L,M)|\alpha|a(0,0)\rangle = C_1(L)\, Y_L^{M*}(\Omega_{\vec{r}}) \; . \tag{22}$$

The reduced matrix element $C_1(L)$ depends on the details of the intrinsic wave functions. Choosing a coordinate system which has its z-axis along $\vec{k}_a$ and the y-axis along $\vec{k}_a \times \vec{k}_b$, substituting from eqs. (13), (17) and (22) into eq. (10) and using the addition theorem for spherical harmonics [20] gives

$$\begin{aligned} T_{ba}(1) = {} & \frac{4\pi}{k_a k_b}(2L+1)^{1/2} C_1(L) \sum_{\ell\ell'} i^{\ell-\ell'} \langle \ell' L 0 0|\ell 0\rangle \\ & \cdot \langle \ell' L, -M M|\ell 0\rangle (2\ell'+1)^{1/2} e^{i[\sigma_\ell(n_a)+\sigma_{\ell'}(n_b)]} \\ & \cdot \beta_{\ell';\ell}(k_b,k_a)\, Y_{\ell'}^{-M}(\theta,0) \; , \end{aligned} \tag{23}$$

where $\beta_{\ell';\ell}$ are the radial integrals

$$\beta_{\ell';\ell} = \int_0^\infty dr\; f_{\ell'}(k_b,r)\, \frac{\partial U}{\partial R_0}\, f_\ell(k_a,r) \; . \tag{24}$$

The product of Clebsch-Gordan coefficients $\langle \ell' L 0 0|\ell 0\rangle\langle \ell' L, -M M|\ell 0\rangle$ results from the integration over the spherical harmonics, and restricts the double summation to

$$\ell = \ell'+L,\ \ell'+L-1,\ \cdots\cdots\ |\ell'-L| \; . \tag{25}$$

A further restriction is imposed by the parity coefficient $\langle \ell' L 0 0|\ell 0\rangle$ which vanishes for odd values of $\ell+\ell'+L$.

Most of the effort in a DWBA calculation goes into evaluating the radial integrals. For the optical potential U the Woods-Saxon shape is often used:

$$U = -(V + iW)\left[1 + \exp\left(\frac{r - R_0}{a}\right)\right]^{-1} . \tag{26}$$

The parameters V, W, R_0 and $\underset{\sim}{a}$ are determined by fitting the elastic scattering angular distribution. These parameter values are then used to generate the f_ℓ contained in the radial integrals. The inelastic cross section is now fixed, except for the reduced matrix element $C_1(L)$ which is obtained by normalization.

III. AUSTERN-BLAIR THEORY

The Austern-Blair theory [12] expresses the inelastic scattering amplitudes in terms of derivatives of the elastic scattering reflection coefficients, whereby the close relation between inelastic and elastic scattering is clarified.

1. Single excitation.

It is assumed that the incident energy is sufficiently high so that

$$\beta_{\ell',\ell}(k_b, k_a) \approx \beta_{\ell',\ell}(k, k) \quad , \tag{27}$$

where $k_b \approx k_a = k$. Further, for $L \ll \ell, \ell'$ it is assumed that

$$\beta_{\ell',\ell}(k, k) \approx \beta_{\bar{\ell},\bar{\ell}}(k, k) \tag{28}$$

where

$$\bar{\ell} = \frac{1}{2}(\ell + \ell') \quad . \tag{29}$$

The latter approximation was shown to be particularly reliable for strongly absorbed particles at high bombarding energies.

Consider the scattering of a particle from two potentials U and $\hat{U}$ which differ only with respect to their radii, ie.

$$\hat{U} = U(r-R_0) + \Delta R_0 \frac{\partial U}{\partial R_0} + \frac{(\Delta R_0)^2}{2!} \frac{\partial^2 U}{\partial R_0^2} + \cdots \quad . \tag{30}$$

By using eq. (18) it can be shown that

$$\frac{d}{dr}\left(\hat{f}_\ell \frac{df_\ell}{dr} - f_\ell \frac{d\hat{f}_\ell}{dr} \right) = \frac{2\mu}{\hbar^2} (U - \hat{U}) f_\ell \hat{f}_\ell \; .$$

Integration on both sides and using the boundary conditions (19) and eq. (30) yields

$$\hat{\eta}_\ell - \eta_\ell = - \frac{i2k}{E_{c.o.m.}} \int_0^\infty dr \left[\Delta R_0 \frac{\partial U}{\partial R_0} + \frac{(\Delta R_0)^2}{2!} \frac{\partial^2 U}{\partial R_0^2} \right] f_\ell \hat{f}_\ell \; ,$$

which leads to

$$\frac{\partial \eta_\ell}{\partial R_0} = -i\,\frac{2k}{E_{c.o.m.}}\int_0^\infty dr\; f_\ell^2(k,r)\,\frac{\partial U}{\partial R_0}\;. \tag{31}$$

For strongly absorbed particles and sufficiently high energies the η_ℓ can be considered as a smooth, continuous function of ℓ of the form

$$\eta_\ell = \eta\,(\ell-\ell_0) \quad , \quad Re\,\eta_{\ell_0} = \tfrac{1}{2} \;, \tag{32}$$

where ℓ_0 is the orbital angular momentum corresponding to the classical grazing trajectory.

$$\ell_0 + \tfrac{1}{2} = \left[(kR)^2 - 2\,n\,kR \right]^{1/2} \;. \tag{33}$$

The radius R is written as the sum of the radii of the projectile and target nucleus. Under the assumptions $\Delta R = \Delta R_0$ and $kR \gg n$, it follows from eqs. (31-33) and eq. (28) that

$$\beta_{\ell',\ell}(k,k) = -\,i\,\frac{E_{c.o.m.}}{2}\,\frac{\partial \eta_{\bar\ell}}{\partial \bar\ell} \tag{34}$$

whereby the radial integrals have been related to the reflection coefficients of the elastic channel.

Defining the scattering amplitude

$$f^{(1)}_{LM}(\theta) = -\frac{\mu}{2\pi\hbar^2}\,T_{ba}(1) \;,\quad k_b = k_a \;, \tag{35}$$

the differential cross section becomes

$$\frac{d\sigma}{d\Omega}(0\rightarrow L) = \sum_{M=-L}^{L} \left| f^{(1)}_{LM}(\theta) \right|^2 . \tag{36}$$

Substitution from eqs. (27) and (34) into eq. (23) gives

$$f^{(1)}_{LM}(\theta) = \tfrac{1}{2}\, i\,\delta_1(L) \sum_{\ell\ell'} i^{\,\ell-\ell'} \langle \ell' L 0 0 | \ell 0\rangle \langle \ell' L, -M M | \ell 0 \rangle$$

$$\cdot\,(2\ell'+1)^{1/2}\, e^{\,i(\sigma_\ell + \sigma_{\ell'})}\,\frac{\partial \eta_{\bar\ell}}{\partial \bar\ell}\, Y^{-M}_{\ell'}(\theta,0) \;, \tag{37}$$

where

$$\delta_1(L) = (2L+1)^{1/2}\, C_1(L) \tag{38}$$

is the deformation distance. It has been suggested by Blair [9] that $\delta_1(L)$ is the quantity which should be determined from experiment.

In eq. (37) the only other unknown quantity is the η_ℓ function. Experience has shown that simple analytic models of the η_ℓ, characterised by a few adjustable parameters, can give a satisfactory description of elastic scattering data for strongly absorbed projectiles [21-22]. In terms of these models it is easy to calculate the corresponding inelastic amplitude defined above.

It must be borne in mind that eq. (37) applies to relatively small excitation energies. As has been indicated [8] there is a reduction in intensity for $k_b < k_a$ and a tendency for the minima in the oscillations to fill up. The most notable effect is, however, a decrease in the period of the diffraction oscillations. These effects are, of course, not contained in the adiabatic AB theory. However, since most of the strongly enhanced quadrupole and octupole levels are lowlying, there are many cases in which eq. (37) can be applied.

In order to investigate the implications of assumption (28), we define the amplitude

$$a_{\ell'}^{LM} = \sum_{\ell=|\ell'-L|}^{\ell'+L} i^{\ell-\ell'-L} \langle \ell' L 0 0 | \ell 0 \rangle \langle \ell' L, -M M | \ell 0 \rangle e^{i(\sigma_\ell + \sigma_{\ell'})} \beta_{\ell'\ell}(k,k) \qquad (39)$$

contained in eq. (23). This provides a measure of the importance of the M-contribution to the angular distribution. By using asymptotic values [23] for the Clebsch-Gordon co-

compare the AB theory to DWBA. The optical model parameters refer to the scattering of 64.3 MeV alpha particles by ^{58}Ni; their values are : $V = 45$ MeV, $W = 20.9$ MeV, $R_o = 6.1$ fm, $a_v = 0.57$ fm, $a_w = 0.58$ fm [25]. For the AB amplitudes, $\beta_{\bar{\ell},\bar{\ell}}$ has been used instead of $-\frac{1}{2} i\, E_{c.o.m}\, \frac{\partial \eta_{\bar{\ell}}}{\partial \bar{\ell}}$. The coefficients $a_{\ell'}^{2M}$ are compared in fig.1. The angular distributions for multipolarities 0 through 6 are shown in figs. 2 and 3. The AB angular distribution is normalized relative to the corresponding DWBA distribution at forward angles. The relative deformation distances are listed in table 1.

Table 1

Comparison of deformation distances in the DWBA and AB theory

L	L/ℓ_0	δ_1 (fm) DWBA	δ_1 (fm) AB
0	0	1	1.00
1	0.05	1	0.95
2	0.10	1	0.88
3	0.15	1	0.80
4	0.20	1	0.72
5	0.25	1	0.71
6	0.30	1	0.68

This example illustrates the remarks of the previous paragraphs. In the case of $L=2$ the AB deformation distance is smaller by 12%. For $L=3$ this discrepancy is quite large, namely 20%. The AB theory agrees very well with DWBA at forward angles (after normalization of course) for these two multipolarities. At extreme forward angles the agreement is less satisfactory. Towards larger angles the AB cross section is too large by approximately a factor two. It is of interest to note that the phase of the oscillations is correctly given by the AB theory.

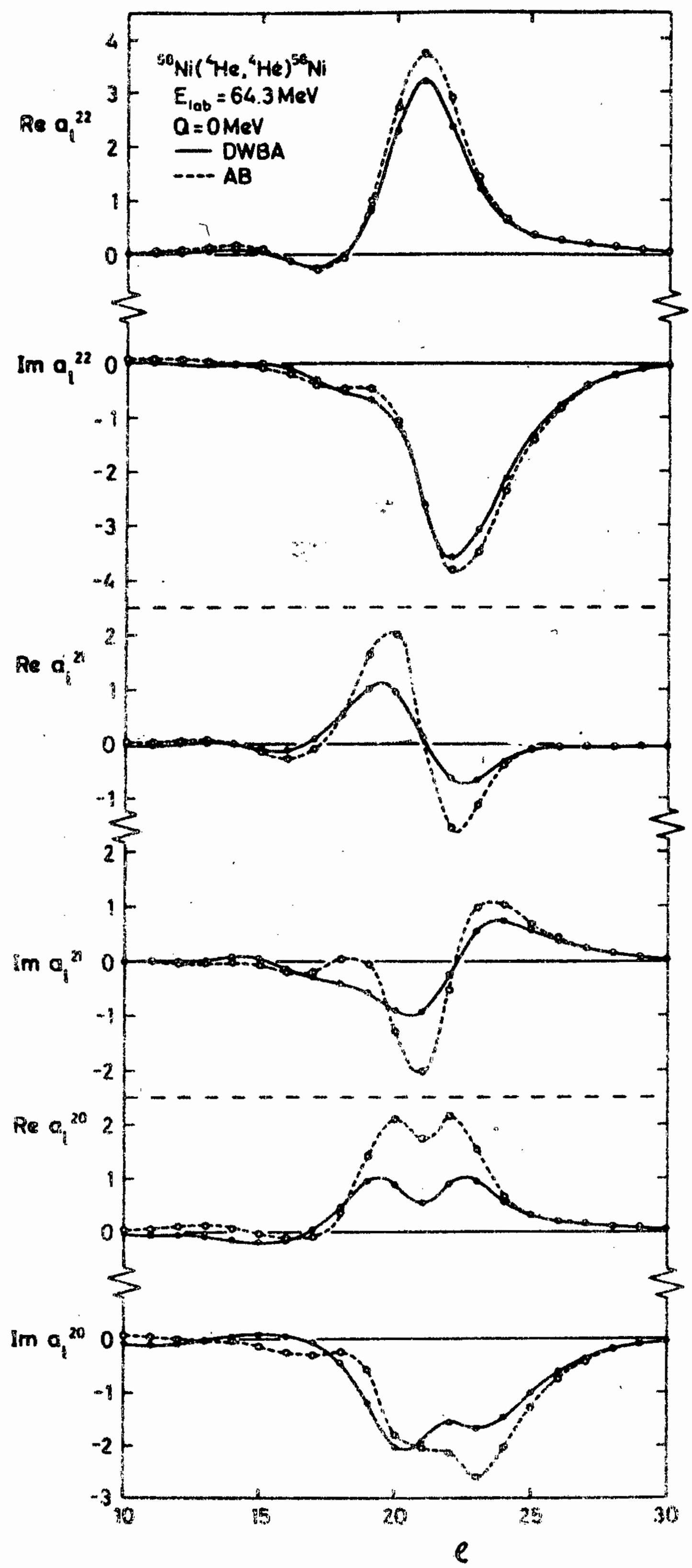

Fig.1. Calculated values for the $a_{\ell'}^{2M}$ coefficients in the DWBA and AB theory, for the optical model parameters given on page 14.

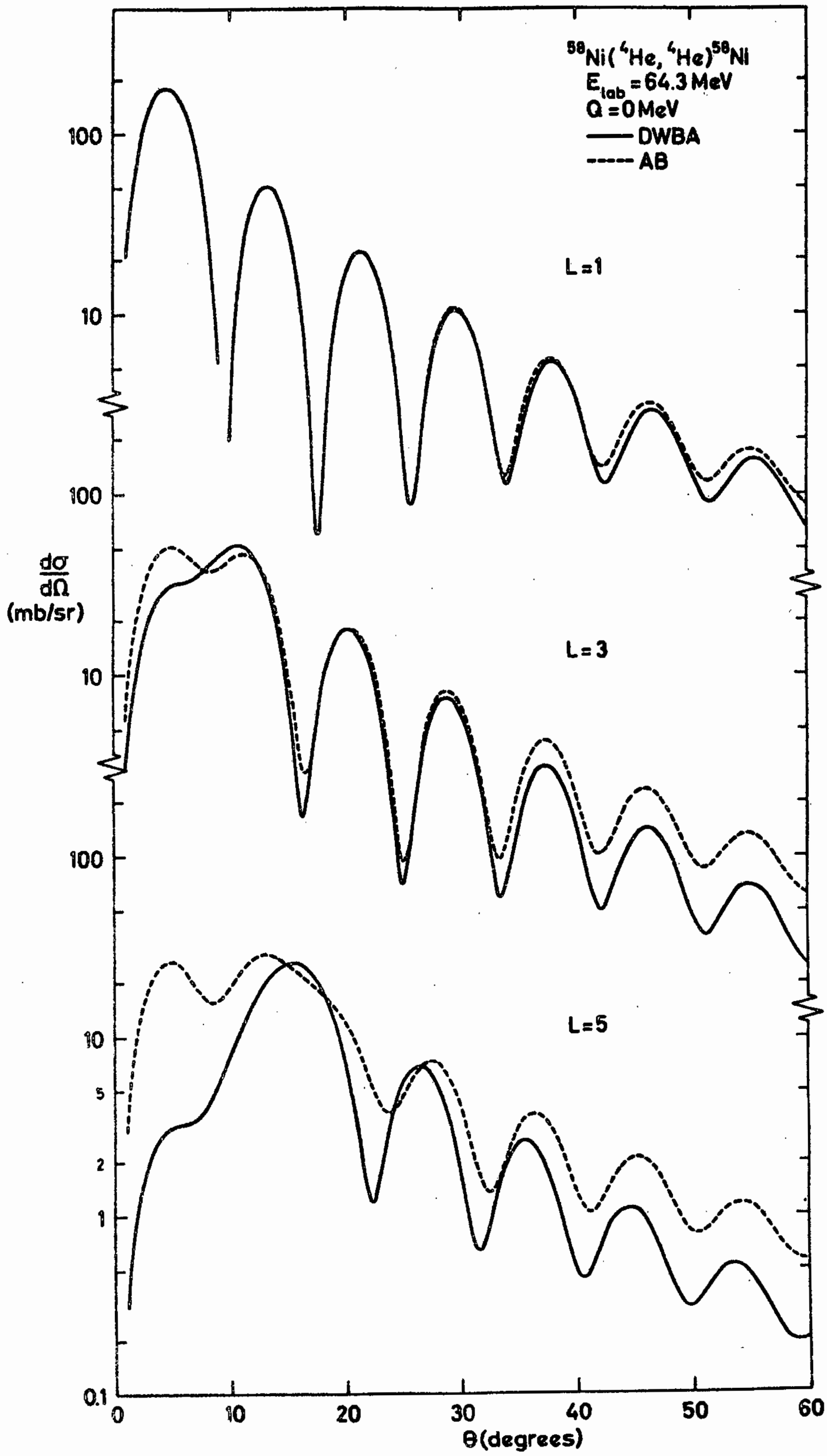

Fig. 2. Calculated odd-parity inelastic angular distributions in the DWBA and AB theory, for the optical model parameters given on page 14. The deformation distances are listed in table 1.

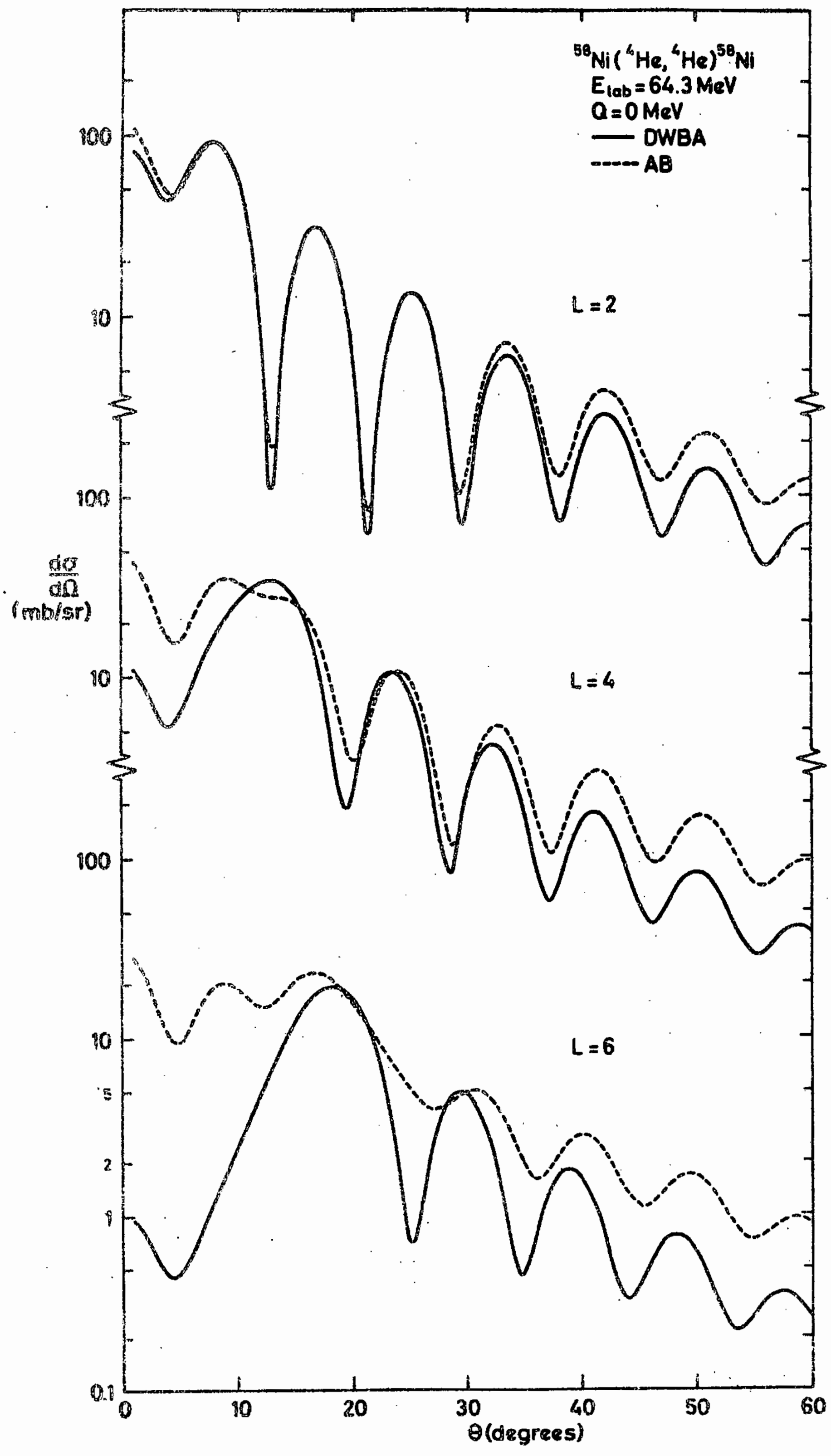

Fig.3. Calculated even-parity angular distributions in the DWBA and AB theory, for the optical model parameters given on page 14. The deformation distances are listed in table 1.

As far as scattering angular distributions are concerned, the AB theory is therefore quite successful. However, the assumptions made about the radial integrals become more important for the study of angular correlation functions. Since the AB theory neglects the Q-value of the reaction, it gives an incorrect prediction for these functions [26]. Slight modifications of the AB assumptions have made it possible to describe the main features of the angular correlation functions [27]. At the same time better agreement is obtained for the deformation parameter.

2. <u>Double excitation</u>

It has been observed [28] that certain inelastic angular distributions show a phase reversal compared to the normal phase [2] exhibited by the majority of inelastic data. This phenomenon can be understood [12, 19, 29] in terms of the second order amplitude (11), which can be written in the form (ref. (12))

$$\begin{aligned} T_{ba}(2) = \Big\langle b(\xi_1) \Big| \iint d\vec{r}_1 \, d\vec{r}_2 \; \chi^{(-)*}(\vec{k}_b, \vec{r}_1) \\ \cdot \Big\{ \frac{1}{2} \frac{\partial^2 U(1)}{\partial R_0^2} \, \alpha^2(\vec{r}_1, \xi_1) \, \delta(\vec{r}_1 - \vec{r}_2) \, \delta(\xi_1 - \xi_2) \\ + \alpha(\vec{r}_1, \xi_1) \frac{\partial U(1)}{\partial R_0} \, G(\vec{r}_1, \vec{r}_2, \xi_1, \xi_2) \\ \cdot \alpha(\vec{r}_2, \xi_2) \frac{\partial U(2)}{\partial R_0} \Big\} \chi^{(+)}(\vec{k}_a, \vec{r}_2) \Big| a(\xi_2) \Big\rangle . \end{aligned} \tag{44}$$

The adiabatic approximation is now introduced in the intermediate Green's function:

$$G = \frac{1}{2} \left(E - T - U - \epsilon_b + i\eta \right)^{-1} + \frac{1}{2} \left(E - T - U - \epsilon_a + i\eta \right)^{-1} \tag{45}$$

where ϵ_a and ϵ_b are the eigenenergies of the nucleus in the initial and final states. Since this approximate Green's function does not depend on the internal coordinates ξ , it may be written in the form

$$G(\vec{r}_1,\vec{r}_2,\xi_1,\xi_2)=\tfrac{1}{2}\left[\,G(k_b,\vec{r}_1,\vec{r}_2)+G(k_a,\vec{r}_1,\vec{r}_2)\right]\delta(\xi_1-\xi_2). \tag{46}$$

The adiabatic amplitude becomes

$$\begin{aligned}T_{ba}(2)\approx \tfrac{1}{2}\langle b/ \iint d\vec{r}_1 d\vec{r}_2\ & \chi^{(-)*}(\vec{k}_b,\vec{r}_1)\\ \cdot\Big\{ & \frac{\partial^2 U(1)}{\partial R_0^2}\,\alpha^2(\vec{r}_1,\xi)\,\delta(\vec{r}_1-\vec{r}_2)\\ & +\alpha(\vec{r}_1,\xi)\frac{\partial U(1)}{\partial R_0}\left[G(k_b,\vec{r}_1,\vec{r}_2)+G(k_a,\vec{r}_1,\vec{r}_2)\right]\\ & \cdot\,\alpha(\vec{r}_2,\xi)\,\frac{\partial U(2)}{\partial R_0}\Big\}\,\chi^{(+)}(\vec{k}_a,\vec{r}_2)\Big/a\rangle.\end{aligned} \tag{47}$$

A further approximation [12] is made by assuming that $\alpha(\vec{r}_1,\xi)$ or $\alpha(\vec{r}_2,\xi)$ may be commuted with the intermediate Green's function to give

$$\begin{aligned}T_{ba}(2)\approx \tfrac{1}{2}C_2(L)\iint d\vec{r}_1 d\vec{r}_2\ & \chi^{(-)*}(\vec{k}_b,\vec{r}_1)\\ \cdot\Big\{ & \frac{\partial^2 U(1)}{\partial R_0^2}\,Y_L^{M*}(\Omega_{\vec{r}_1})\,\delta(\vec{r}_1-\vec{r}_2)\\ & +\frac{\partial U(1)}{\partial R_0}\left[G(k_b,\vec{r}_1,\vec{r}_2)\,Y_L^{M*}(\Omega_{\vec{r}_2})+Y_L^{M*}(\Omega_{\vec{r}_1})\,G(k_a,\vec{r}_1,\vec{r}_2)\right]\\ & \cdot\,\frac{\partial U(2)}{\partial R_0}\Big\}\,\chi^{(+)}(\vec{k}_a,\vec{r}_2)\ ,\end{aligned} \tag{48}$$

where the reduced matrix element $C_2(L)$ is contained in

$$\langle b(L,M)|\,\alpha^2/a(0,0)\rangle = C_2(L)\,Y_L^{M*}(\Omega_{\vec{r}})\ . \tag{49}$$

Differentiation of eq. (14) with respect to R_0 gives

$$\left(\frac{\hbar^2}{2\mu}\nabla^2+E-U-U_c\right)\frac{\partial\chi}{\partial R_0}=\frac{\partial U}{\partial R_0}\,\chi\ . \tag{50}$$

The inhomogeneous solution of eq. (50) is

$$\frac{\partial\chi}{\partial R_0} = \int d\vec{r}\,' \, G(k,\vec{r},\vec{r}\,') \, \frac{\partial U(r')}{\partial R_0} \, \chi(\vec{k},\vec{r}\,') \; . \tag{51}$$

Inspection of eqs. (48) and (51) shows that

$$T_{ba}(2) = \frac{1}{2} C_2(L) \frac{\partial}{\partial R_0} \int d\vec{r} \, \chi^{(-)*}(\vec{k}_b,\vec{r}) \, Y_L^{M*}(\Omega_{\vec{r}}) \cdot \frac{\partial U(r)}{\partial R_0} \, \chi^{(+)}(\vec{k}_a,\vec{r}) \; . \tag{52}$$

Comparison with the expression for single excitation defined in eq. (10) gives

$$T_{ba}(2) = \frac{1}{2} \frac{C_2(L)}{C_1(L)} \frac{\partial}{\partial R_0} T_{ba}(1) \; . \tag{53}$$

In terms of scattering amplitudes, it follows from eqs. (35), (37) and (53) that

$$f_{LM}^{(2)}(\theta) = \left(-\frac{ik}{4}\right)(2L+1)^{\frac{1}{2}} C_2(L) \sum_{\ell\ell'} i^{\ell-\ell'} \langle \ell' L 0 0 | \ell 0 \rangle \cdot \langle \ell' L, -M M | \ell 0 \rangle (2\ell'+1)^{1/2} \, e^{i(\sigma_\ell + \sigma_{\ell'})} \cdot \frac{\partial^2 \eta_{\bar{\ell}}}{\partial \bar{\ell}^2} \, Y_{\ell'}^{-M}(\theta, 0) \; . \tag{54}$$

In arriving at this result assumptions (32) and (33) have been used. The differential cross section for double excitation is given by

$$\frac{d\sigma}{d\Omega}(0 \to L) = \sum_{M=-L}^{L} \left| f_{LM}^{(2)}(\theta) \right|^2 \; . \tag{55}$$

3. <u>Mutual excitation</u>

In the scattering of heavy ions it is possible that both the projectile and target nucleus are excited. It is therefore of interest to extend the AB formalism to mutual excitation of nuclei. For this purpose, the extended optical potential is assumed to depend on the coordinates of the projectile as well as target nucleus, i.e.

$$R = R_0 + \alpha_1 + \alpha_2 \quad . \tag{56}$$

The same procedure which led to eqs. (10) and (11) yields

$$\begin{aligned} T_{ba}(mut) = \Big\langle b\, \chi^{(-)}(\vec{k}_b, \vec{r}_1) \Big| & \frac{2\alpha_1\alpha_2}{2!} \frac{\partial^2 U(1)}{\partial R_0^2} \\ & + \alpha_1 \frac{\partial U}{\partial R_0} G \alpha_2 \frac{\partial U}{\partial R_0} + \alpha_2 \frac{\partial U}{\partial R_0} G \alpha_1 \frac{\partial U}{\partial R_0} \Big| \\ & a\, \chi^{(+)}(\vec{k}_a, \vec{r}_2) \Big\rangle . \end{aligned} \tag{57}$$

The matrix element $\langle b | 2\alpha_1\alpha_2 | a \rangle$ can be factored by introducing the adiabatic assumption (46) and making suitable commutations of the operators α_1 and α_2. If the spins and projections of the final nuclei are L_1, M_1 and L_2, M_2 respectively, one finds

$$\langle b | 2\alpha_1\alpha_2 | a \rangle = 2 C_1(L_1) C_1(L_2) Y_{L_1}^{M_1 *} Y_{L_2}^{M_2 *} . \tag{58}$$

Following the same procedure outlined for double excitation gives

$$\begin{aligned} f_{L_1 M_1 L_2 M_2}(\theta) = \left(-\frac{ik}{4}\right) \pi^{-1/2} S_1(L_1) S_1(L_2) \sum_L \langle L_1 L_2 00 | L0 \rangle \\ \cdot \langle L_1 L_2 M_1 M_2 | LM \rangle \sum_{\ell\ell'} i^{\ell-\ell'} \\ \cdot \langle \ell' L 00 | \ell 0 \rangle \langle \ell' L, -M M | \ell 0 \rangle (2\ell'+1)^{1/2} \\ \cdot e^{i(\sigma_\ell + \sigma_{\ell'})} \frac{\partial^2 \eta_{\bar{\ell}}}{\partial \bar{\ell}^2} Y_{\ell'}^{-M}(\theta, 0) . \end{aligned} \tag{59}$$

The differential cross section for mutual excitation of multipolarities L_1 and L_2 becomes

$$\frac{d\sigma}{d\Omega} = \sum_{M_1 M_2} \left| f_{L_1 M_1 L_2 M_2}(\theta) \right|^2 . \tag{60}$$

By using an orthogonality relation [20] satisfied by Clebsch-Gordan coefficients, it can be shown that eq. (60) reduces to

$$\frac{d\sigma}{d\Omega} = \sum_{LM} \left| f_{LM}^{mut}(\theta) \right|^2 \tag{61}$$

where

$$f_{LM}^{mut}(\theta) = \left(\frac{-ik}{4}\right) \pi^{-1/2} \langle L_1 L_2 00 | L0 \rangle \delta_1(L_1) \delta_1(L_2) \cdot \sum_{\ell\ell'} i^{\ell-\ell'} \langle \ell' L 00 | \ell 0 \rangle \langle \ell' L, -MM | \ell 0 \rangle \cdot (2\ell'+1)^{1/2} e^{i(\sigma_\ell + \sigma_{\ell'})} \frac{\partial^2 \eta_{\bar{\ell}}}{\partial \bar{\ell}^2} Y_{\ell'}^{-M}(\theta, 0) . \tag{62}$$

This expression is completely determined by the parameters for single excitation.

4. Identical particles

If a reaction channel contains a pair of identical particles, the wave functions must be properly symmetrized. For single excitation the scattering amplitude is [30]

$$f_{LM,S}^{(1)}(\theta) = \frac{-\mu}{2\pi k^2} \Big[\langle b\, \chi^{(-)}(\vec{k}_b, \vec{r}_1) | \alpha \frac{\partial U}{\partial R_0} | a\, \chi^{(+)}(\vec{k}_a, \vec{r}_2) \rangle + \langle b\, \chi^{(-)}(\vec{k}_b, \vec{r}_1) | \alpha \frac{\partial U}{\partial R_0} | a\, \chi^{(+)}(\vec{k}_a, -\vec{r}_2) \rangle \Big] . \tag{63}$$

Partial wave expansions can be carried out as outlined in Section II for non-identical particles. The result is

$$f_{LM,S}^{(1)}(\theta) = \frac{1}{2} i\, \delta_1(L) \sum_{\ell\ell'} i^{\ell-\ell'} \langle \ell' L 00 | \ell 0 \rangle \langle \ell' L, -MM | \ell 0 \rangle \cdot (2\ell'+1)^{1/2} e^{i(\sigma_\ell + \sigma_{\ell'})} \frac{\partial \eta_{\bar{\ell}}}{\partial \bar{\ell}} [1 + (-)^\ell] Y_{\ell'}^{-M}(\theta, 0)$$

where the factor $(-)^\ell$ comes from the parity property of spherical harmonics. Since $\langle \ell' L 0 0 | \ell 0 \rangle$ vanishes for odd values of $\ell + \ell' + L$ it is possible to write the above result in the form

$$f^{(1)}_{LM,S}(\theta) = f^{(1)}_{LM}(\theta) + (-)^{L+M} f^{(1)}_{LM}(\pi - \theta) \quad . \tag{64}$$

The differential cross section is given by

$$\frac{d\sigma}{d\Omega}(0 \to L) = \sum_{M=-L}^{L} \left| f^{(1)}_{LM,S}(\theta) \right|^2 . \tag{65}$$

A similar result holds for double excitation:

$$\frac{d\sigma}{d\Omega}(0 \to L) = \sum_{M=-L}^{L} \left| f^{(2)}_{LM,S}(\theta) \right|^2 \tag{66}$$

where

$$f^{(2)}_{LM,S}(\theta) = f^{(2)}_{LM}(\theta) + (-)^{L+M} f^{(2)}_{LM}(\pi - \theta) \quad . \tag{67}$$

In the case of mutual excitation both the incident and exit channels contain identical particles. The symmetrized scattering amplitude becomes [30]

$$\begin{aligned} f_{L_1 M_1 L_2 M_2, S}(\theta) = {} & -\frac{\mu}{2\pi\hbar^2} \Big\langle b \tfrac{1}{\sqrt{2}} \left[\chi^{(-)}(\vec{k}_b, \vec{r}_1) + \chi^{(-)}(\vec{k}_b, -\vec{r}_1) \right] \Big| \\ & + \frac{2\alpha_1\alpha_2}{2!} \frac{\partial^2 U}{\partial R_0^2} + \alpha_1 \frac{\partial U}{\partial R_0} G \alpha_2 \frac{\partial U}{\partial R_0} \\ & + \alpha_2 \frac{\partial U}{\partial R_0} G \alpha_1 \frac{\partial U}{\partial R_0} \Big| a \tfrac{1}{\sqrt{2}} \left[\chi^{(+)}(\vec{k}_a, \vec{r}_2) \right. \\ & \left. + \chi^{(+)}(\vec{k}_a, -\vec{r}_2) \right] \Big\rangle \end{aligned} \tag{68}$$

which leads to

$$\frac{d\sigma}{d\Omega}\begin{pmatrix} 0 \to L_1 \\ 0 \to L_1 \end{pmatrix} = \sum_{LM} \left| f^{mut}_{LM,S}(\theta) \right|^2 , \tag{69}$$

where

$$f_{LM,S}^{mud}(\theta) = f_{LM}^{mud}(\theta) + (-)^{L+M} f_{LM}^{mud}(\pi-\theta) . \qquad (70)$$

IV. EVALUATION IN CLOSED FORM

In the strong absorption model formalism [10] for elastic scattering, closed formulae are derived for the cross section under very general assumptions about the reflection coefficients. For composite particles it is assumed that the η_ℓ coefficients vary smoothly as a continuous function of ℓ. The only essential requirements are that $\frac{d\eta_\ell}{d\ell}$ should possess a simple Fourier transform, and that η_ℓ depends on the cutoff parameter ℓ_0 and the rounding parameter Δ in the combination

$$\eta_\ell = \eta\left(\frac{\ell-\ell_0}{\Delta}\right) \quad . \tag{71}$$

The assumptions of continuity and of strong absorption imply the conditions

$$\ell_0 \gg 1 \quad , \quad \Delta \ll \ell_0 \quad . \tag{72}$$

Since the AB amplitudes contain the derivatives of the η_ℓ, the SAM formalism can be extended [13] to inelastic scattering and the connection with elastic scattering becomes more explicit.

1. Derivation of closed expressions

The scattering amplitude for single excitation is

$$f^{(1)}_{LM}(\theta) = \tfrac{1}{2} i\, \delta_1(L) \sum_{\ell\ell'} i^{\ell-\ell'} \langle \ell' L 0 0 | \ell 0\rangle \cdot \langle \ell' L, -M M | \ell 0\rangle (2\ell'+1)^{1/2}\, e^{i(\sigma_\ell + \sigma_{\ell'})} \cdot \frac{\partial \eta_{\bar\ell}}{\partial \bar\ell}\, Y^{-M}_{\ell'}(\theta, 0) \quad , \tag{37}$$

subject to assumptions (71) and (72) and the condition

$$L \ll \ell_0 \; . \tag{73}$$

Because of assumption (72), $\frac{\partial n_{\bar{\ell}}}{\partial \bar{\ell}}$ is confined to a narrow range of $\bar{\ell}$ – values in the vicinity of ℓ_0. This makes it possible to introduce the approximation

$$\sigma_{\ell} + \sigma_{\ell'} \approx 2\sigma_T - \tfrac{1}{2}\theta_c + (t-T)\theta_c \; , \tag{74}$$

where

$$T = \ell_0 + \tfrac{1}{2} \quad , \quad t = \bar{\ell} + \tfrac{1}{2} \tag{75}$$

and the critical angle θ_c is defined by [10]

$$\theta_c = 2 \arctan\left(\frac{n}{T} \right) . \tag{76}$$

Expression (74) is derived in Appendix II and holds for Coulomb parameters $n \lesssim T$. The amplitude (37) now becomes

$$f^{(1)}_{LM}(\theta) \approx \tfrac{1}{2} i \; e^{i(2\sigma_T - \frac{1}{2}\theta_c)} \, S_1(L) \sum_{\ell\ell'} i^{\ell-\ell'} \cdot \langle \ell' L 0 0 | \ell 0 \rangle \langle \ell' L, -M M | \ell 0 \rangle \, (2\ell'+1)^{1/2} \cdot \rho(\bar{\ell}) \; Y^{-M}_{\ell'}(\theta, 0) \; , \tag{77}$$

where

$$\rho(\bar{\ell}) = e^{i(t-T)\theta_c} \, \frac{\partial n_{\bar{\ell}}}{\partial \bar{\ell}} \; . \tag{78}$$

Taylor expansion of $\rho(\bar{\ell})$ about ℓ' gives

$$f^{(1)}_{LM}(\theta) = \tfrac{1}{2} i^{L+1} \, e^{i(2\sigma_T - \frac{1}{2}\theta_c)} \, S_1(L) \sum_{f=0}^{\infty} \sum_{\ell'=0}^{\infty} \cdot C^{(f)}_{LM}(\ell') \, (2\ell'+1)^{1/2} \, \rho^{(f)}(\ell') \, Y^{-M}_{\ell'}(\theta, 0) \, , \tag{79}$$

where

$$C_{LM}^{(r)}(\ell') = \sum_{\ell=|\ell'-L|}^{\ell'+L} i^{\ell-\ell'-L} \langle \ell' L 0 0 | \ell 0 \rangle \langle \ell' L, -M M | \ell 0 \rangle \cdot \frac{1}{r!} \left(\frac{\ell-\ell'}{2} \right)^r , \tag{80}$$

and $\rho^{(r)}(\ell')$ denotes the r-th derivative of $\rho(\ell')$.

Since $\rho^{(r)}(\ell')$ is localised in the vicinity of ℓ_0 and because of assumption (73), the contributions to (79) for $\ell' < L$ may be neglected, so that the ℓ- summation extends from $\ell'-L$ to $\ell'+L$. Furthermore, under these conditions the Clebsch-Gordan coefficient $\langle \ell' L, -M M | \ell 0 \rangle$ is a slowly varying function of ℓ' for a given $\ell - \ell'$, L and M. Since the main contributions to (80) come from $\ell' \approx \ell_0$, it is possible to approximate $C_{LM}^{(r)}(\ell') \approx C_{LM}^{(r)}(\ell_0)$, where

$$C_{LM}^{(r)}(\ell_0) = \sum_{\lambda=-L}^{L} i^{\lambda-L} \langle \ell_0 L 0 0 | \ell_0+\lambda, 0 \rangle \langle \ell_0 L, -M M | \ell_0+\lambda, 0 \rangle \cdot \frac{1}{r!} \left(\tfrac{1}{2}\lambda \right)^r \tag{81}$$

are real coefficients. The nearest integer to ℓ_0 must be used in eq. (81). Equation (79) now becomes

$$f_{LM}^{(1)}(\theta) = \tfrac{1}{2} i^{L+1} e^{i(2\sigma_T - \frac{1}{2}\theta_c)} \delta_1(L) \sum_{r=0}^{\infty} C_{LM}^{(r)}(\ell_0)\, S_M^{(r)} \tag{82}$$

where

$$S_M^{(r)} = \sum_{\ell'=0}^{\infty} (2\ell'+1)^{1/2} \rho^{(r)}(\ell')\, Y_{\ell'}^{-M}(\theta, 0) \, . \tag{83}$$

The spherical harmonics are approximated by the asymptotic form

$$Y_{\ell'}^{-M}(\theta,0) \approx (-)^{\frac{1}{2}(M-|M|)} \left[\frac{2\ell'+1}{4\pi} \frac{(\ell'-|M|)!}{(\ell'+|M|)!} \right]^{1/2} (\ell'+\tfrac{1}{2})^{|M|} \cdot \left(\frac{\theta}{\sin\theta} \right)^{1/2} J_{|M|}\left[(\ell'+\tfrac{1}{2})\theta \right] , \tag{84}$$

where $J_{|M|}$ is the cylindrical Bessel function; this asymptotic form is justified in Appendix III; it is valid under the conditions $\pi - \theta \gg 1/(4T)$, $|M| \ll \ell$. Substitution of (84) in eq. (83) gives

$$S_M^{(r)} = (-)^{\frac{1}{2}(M-|M|)} \pi^{-\frac{1}{2}} (\theta/\sin\theta)^{\frac{1}{2}} \sum_{\ell=0}^{\infty} \left[\frac{(\ell-|M|)!}{(\ell+|M|)!} \right]^{\frac{1}{2}} \cdot (\ell+\tfrac{1}{2})^{|M|+1} \rho^{(r)}(\ell) J_{|M|}[(\ell+\tfrac{1}{2})\theta] .$$

Since $\ell_0 \gg |M|$ and $\rho^{(r)}(\ell)$ is localised in the vicinity of ℓ_0, it follows

$$S_M^{(r)} \approx (-)^{\frac{1}{2}(M-|M|)} \pi^{-\frac{1}{2}} (\theta/\sin\theta)^{\frac{1}{2}} (\ell_0+\tfrac{1}{2})^{|M|} I_M^{(r)}, \tag{85}$$

where

$$I_M^{(r)} = \sum_{\ell=0}^{\infty} (\ell+\tfrac{1}{2})^{-|M|+1} \rho^{(r)}(\ell) J_{|M|}[(\ell+\tfrac{1}{2})\theta] . \tag{86}$$

A procedure similar to the one developed by Venter [31] will now be followed to evaluate the sum in eq. (86). Applying the Poisson summation formula gives

$$I_M^{(r)} = \sum_{s=-\infty}^{\infty} (-)^s \int_{-\infty}^{\infty} dt \, [e^{-i2\pi s t} \rho^{(r)}(t)][t^{-|M|+1} J_{|M|}(t\theta)] , \tag{87}$$

Integrating by parts and using the relation

$$\frac{d}{dz}[z^{-\nu} J_\nu(z)] = -z^{-\nu} J_{\nu+1}(z) , \tag{88}$$

leads to

$$I_M^{(r)} = \frac{1}{\theta} \sum_{s=-\infty}^{\infty} (-)^s \int_{-\infty}^{\infty} dt \, e^{-i2\pi s t} [\rho^{(r+1)}(t) - i2\pi s \, \rho^{(r)}(t)] \cdot t^{-|M|+1} J_{|M|-1}(t\theta) \tag{89}$$

since $\rho^{(r)}(t) \to 0$ if $t \to \pm\infty$.

Now we use an integral representation of the Bessel function [32]

$$J_\nu(x) = \frac{1}{2\pi i} \frac{\Gamma(\frac{1}{2}-\nu)}{\Gamma(\frac{1}{2})} \left(\tfrac{1}{2}x\right)^\nu \int^{(1^+,-1^-)} dz\; e^{ixz} (z^2-1)^{\nu-\frac{1}{2}} .$$

$$(\nu+\tfrac{1}{2} \neq -1,-2,---) \tag{90}$$

Substituting into eq. (89) and changing the order of integration gives

$$I_M^{(r)} = \frac{1}{i\pi} \frac{\Gamma(\frac{3}{2}-|M|)}{\Gamma(\frac{1}{2})} \frac{(\frac{1}{2}\theta)^{|M|}}{\theta^2} \sum_{s=-\infty}^{\infty} (-)^s \int^{(1^+,-1^-)}$$

$$dz\; e^{i(\theta z - 2\pi s)T} (z^2-1)^{|M|-3/2}\; B_{rs}(\Delta\theta z), \tag{91}$$

where

$$B_{rs}(\Delta\theta z) = \frac{\theta z}{\theta z - 2\pi s} \int_{-\infty}^{\infty} dt\; e^{i[(\theta z-2\pi s)(t-T)]} \rho^{(r+1)}(t). \tag{92}$$

This function can be written in the form

$$B_{rs}(\Delta\theta z) = D_{rs}^{(1)}(\Delta\theta z) + \Delta\theta z\, D_{rs}^{(2)}(\Delta\theta z) \tag{93}$$

where

$$D_{rs}^{(1)}(\Delta\theta z) = \tfrac{1}{2}\left[\, B_{rs}(\Delta\theta z) + B_{rs}(-\Delta\theta z)\right]$$

$$D_{rs}^{(2)}(\Delta\theta z) = \tfrac{1}{2}(\Delta\theta z)^{-1}\left[\, B_{rs}(\Delta\theta z) - B_{rs}(-\Delta\theta z)\right] \tag{94}$$

are even functions of z. Therefore a Taylor expansion of these functions about z=1 can be written in the form

$$D_{rs}^{(1,2)}(\Delta\theta z) = \sum_{q=0}^{\infty} \frac{(\Delta\theta)^{2q}}{q!} \frac{\partial^q D_{rs}^{(1,2)}(\Delta\theta)}{\partial[(\Delta\theta)^2]^q} (z^2-1)^q \,, \tag{95}$$

and eq. (91) becomes

$$I_M^{(r)} = \frac{1}{\theta} T^{1-|M|} \sum_{s=-\infty}^{\infty} (-)^s e^{-i2\pi sT} \sum_{q=0}^{\infty} \frac{\Gamma(|M|-\frac{1}{2}+q)}{q!\,\Gamma(|M|-\frac{1}{2})}$$

$$\cdot \left(\frac{-\Delta}{T}\right)^q (2\Delta\theta)^q \left[\frac{\partial^q D_{rs}^{(1)}(\Delta\theta)}{\partial[(\Delta\theta)^2]^q} J_{|M|-1+q}(T\theta) \right.$$

$$\left. + i\,\Delta\theta \frac{\partial^q D_{rs}^{(2)}(\Delta\theta)}{\partial[(\Delta\theta)^2]^q} J_{|M|+q}(T\theta) \right] . \tag{96}$$

Since $p^{(r+1)}(t)$ is a function of $(t-T)$ and $p^{(r+1)}(t) \to 0$ for $t \to \pm\infty$ it follows from eq. (92) that

$$B_{rs}(\Delta\theta) = -i\theta\left[-i(\theta-2\pi s)\right]^r B_s(\theta) , \tag{97}$$

where

$$B_s(\theta) = \int_{-\infty}^{\infty} e^{i[(\theta-2\pi s)\tau]} p(\tau)\, d\tau \tag{98}$$

and $\tau = t-T$. Substituting from eqs. (94) and (97) into eq. (96) gives

$$I_M^{(r)} = -\frac{i}{2\theta} T^{1-|M|} \sum_{s=-\infty}^{\infty} (-)^s e^{-i2\pi sT} \sum_{q=0}^{\infty} \frac{\Gamma(|M|-\frac{1}{2}+q)}{q!\,\Gamma(|M|-\frac{1}{2})}$$

$$\cdot \left(\frac{-\Delta}{T}\right)^q (2\Delta\theta)^q \left[J_{|M|-1+q}(T\theta) \frac{\partial^q}{\partial[(\Delta\theta)^2]^q} \right.$$

$$\left\{ \theta\left[-i(\theta-2\pi s)\right]^r B_s(\theta) - \theta\left[i(\theta+2\pi s)\right]^r B_s(-\theta) \right\}$$

$$+ i\theta J_{|M|+q}(T\theta) \frac{\partial^q}{\partial[(\Delta\theta)^2]^q} \left\{ \left[-i(\theta-2\pi s)\right]^r B_s(\theta) \right.$$

$$\left. \left. + \left[i(\theta+2\pi s)\right]^r B_s(-\theta) \right\} \right] . \tag{99}$$

Substituting into eq. (82) and carrying out the r-summation gives

$$f^{(1)}_{LM}(\theta) = (-)^{\frac{1}{2}(M-|M|)}\, i^{L}\, e^{i(2\sigma_T - \frac{1}{2}\theta_c)}\, \delta_1(L)\, \frac{T}{4\pi^{1/2}}\, (\theta/\sin\theta)^{\frac{1}{2}}$$

$$\cdot \frac{1}{\theta} \sum_{\lambda=-L}^{L} i^{\lambda-L} \langle l_0 L 0 0 | l_0+\lambda, 0\rangle \langle l_0 L, -M M | l_0+\lambda, 0\rangle$$

$$\cdot \sum_{q=0}^{\infty} \frac{\Gamma(|M|-\frac{1}{2}+q)}{q!\,\Gamma(|M|-\frac{1}{2})} \left(\frac{-\Delta}{T}\right)^q (2\Delta\theta)^q$$

$$\cdot \Big[J_{|M|-1+q}(T\theta) \frac{\partial^q}{\partial[(\Delta\theta)^2]^q} \sum_{s=-\infty}^{\infty} (-)^s e^{-i2\pi s(T-\frac{\lambda}{2})}$$

$$\cdot \left\{ e^{-i\lambda\theta/2}\, \theta\, B_s(\theta) - e^{i\lambda\theta/2}\, \theta\, B_s(-\theta) \right\}$$

$$+ i\theta\, J_{|M|+q}(T\theta) \frac{\partial^q}{\partial[(\Delta\theta)^2]^q} \sum_{s=-\infty}^{\infty} (-)^s e^{-i2\pi s(T-\lambda/2)}$$

$$\cdot \left\{ e^{-i\lambda\theta/2} B_s(\theta) + e^{i\lambda\theta/2} B_s(-\theta) \right\} \Big] . \qquad (100)$$

This form of the scattering amplitude is convenient for further approximations. The s- and q-summations will now be investigated for the special case

$$P(t) = e^{i(t-T)\theta_c} \frac{dg}{dt} , \qquad (101)$$

where $g = [1 + e^{(T-t)/\Delta}]^{-1}$. From eq. (98) it follows

$$B_s(\pm\theta) = \frac{\pi\Delta(\theta_c \pm \theta - 2\pi s)}{\sinh[\pi\Delta(\theta_c \pm \theta - 2\pi s)]} .$$

It is clear that the s-series rapidly converge for $|s| \geq 1$ at all angles. Furthermore, under the condition

$$\frac{\pi\Delta(2\pi-\theta_c-\theta)}{\sinh[\pi\Delta(2\pi-\theta_c-\theta)]} \ll \frac{\pi\Delta(\theta_c+\theta)}{\sinh[\pi\Delta(\theta_c+\theta)]} \tag{102}$$

it is possible to restrict the s-summation to the first term s=o. This amounts to replacing the summation in eq. (86) directly by an integral. Equation (102) shows that this approximation only fails at extreme backward angles for reasonable values of Δ . Therefore eq. (100) becomes

$$f^{(1)}_{LM}(\theta) \approx (-)^{\frac{1}{2}(M-|M|)}\, i^L\, e^{i(2\sigma_T-\frac{1}{2}\theta_c)}\, \delta_1(L)\, \frac{T}{4\pi^{1/2}}\, (\theta/\sin\theta)^{\frac{1}{2}}$$

$$\cdot \frac{1}{\theta} \sum_{\lambda=-L}^{L} i^{\lambda-L} \langle \ell_0 L 0 0 | \ell_0+\lambda, 0\rangle \langle \ell_0 L, -M M | \ell_0+\lambda, 0\rangle$$

$$\cdot \sum_{q=0}^{\infty} \frac{\Gamma(|M|-\frac{1}{2}+q)}{q!\,\Gamma(|M|-\frac{1}{2})} \left(\frac{-\Delta}{T}\right)^q (2\Delta\theta)^q$$

$$\cdot \Big[J_{|M|-1+q}(T\theta) \frac{\partial^q}{\partial[(\Delta\theta)^2]^q} \Big\{ e^{-i\lambda\theta/2}\, \theta\, B_0(\theta) - e^{i\lambda\theta/2}\, \theta\, B_0(-\theta) \Big\}$$

$$+ i\theta\, J_{|M|+q}(T\theta) \frac{\partial^q}{\partial[(\Delta\theta)^2]^q} \Big\{ e^{-i\lambda\theta/2} B_0(\theta) + e^{i\lambda\theta/2} B_0(-\theta) \Big\} \Big] \tag{103}$$

It is noticed from eq. (103) that the q-series contains the strong absorption parameter (72) in the factor $(-\Delta/T)^q$. This suggests approximating the q-summation by the first few terms. The q-series is investigated for the parametrization (101) in Appendix IV. It is shown that this series is well approximated by the first term q=o under the conditions

$$\left|(|M|-\tfrac{1}{2})\frac{\lambda}{2T}\right| \ll 1 \quad , \quad \text{all } \theta$$

$$\left|(|M|-\tfrac{1}{2})\frac{\pi\Delta}{T}\left[\frac{1}{\pi\Delta\theta} \pm \frac{1}{\pi\Delta(\theta_c\pm\theta)} \mp \coth\{\pi\Delta(\theta_c\pm\theta)\}\right]\right| \ll 1 \quad , \; T\theta \gg 1. \tag{104}$$

These are the strong absorption conditions. The second condition is satisfied for not too large angles.

The final expression becomes

$$f_{LM}^{(1)}(\theta) \approx (-)^{\frac{1}{2}(M-|M|)}\, i^{L+1}\, e^{i(2\sigma_T-\frac{1}{2}\theta_c)}\, \delta_1(L)\frac{T}{4\pi^{1/2}}(\theta/\sin\theta)^{1/2}$$
$$\cdot\Big\{\left[H_-(\theta)+H_+(\theta)\right]\left[\alpha_{LM}(\theta)\,J_{|M|}(T\theta)-\beta_{LM}(\theta)\,J_{|M|-1}(T\theta)\right]$$
$$+i\left[H_-(\theta)-H_+(\theta)\right]\left[\alpha_{LM}(\theta)\,J_{|M|-1}(T\theta)+\beta_{LM}(\theta)\,J_{|M|}(T\theta)\right]\Big\} \tag{105}$$

where

$$\alpha_{LM}(\theta)+i\beta_{LM}(\theta)=\sum_{\lambda=-L}^{L} i^{\lambda-L}\langle \ell_0 L 0 0|\ell_0+\lambda,0\rangle\langle \ell_0 L,-MM|\ell_0+\lambda,0\rangle \cdot e^{i\lambda\theta/2} \tag{106}$$

The form factors are defined by

$$H_\pm(\theta)=B_0(\pm\theta)=\int_{-\infty}^{\infty} e^{i\phi_\pm\tau}\,\frac{\partial\eta(\tau)}{\partial\tau}\,d\tau \tag{107}$$

where

$$\phi_\pm=\theta_c\pm\theta\ . \tag{108}$$

The same procedure can be followed for the double and mutual excitation amplitudes given by eqs. (54) and (62), respectively. Apart from reduced matrix elements and numerical factors, the only difference is that the function $P(\bar{\varepsilon})$ now takes the form

$$P(\bar{\varepsilon})=e^{i(t-T)\theta_c}\,\frac{\partial^2\eta_{\bar{\varepsilon}}}{\partial\bar{\varepsilon}^2}\ . \tag{109}$$

This introduces the form factor

$$i\,\phi_\pm H_\pm=\int_{-\infty}^{\infty} e^{i\phi_\pm\tau}\,\frac{\partial^2\eta(\tau)}{\partial\tau^2}\,d\tau\ . \tag{110}$$

The resulting amplitudes are given by

$$f_{LM}^{(3)}(\theta) = (-)^{\frac{1}{2}(M-|M|)}\, i^{L+1}\, e^{i(2\sigma_T - \frac{1}{2}\theta_c)} (2L+1)^{\frac{1}{2}}\, C_3(L)$$
$$\cdot(-ik/2)\,[T/(4\pi^{\frac{1}{2}})]\,(\theta/\sin\theta)^{\frac{1}{2}} \Big\{ [\phi_- H_-(\theta) + \phi_+ H_+(\theta)]$$
$$\cdot [\alpha_{LM}(\theta) J_{|M|}(T\theta) - \beta_{LM}(\theta) J_{|M|-1}(T\theta)]$$
$$+ i[\phi_- H_-(\theta) - \phi_+ H_+(\theta)][\alpha_{LM}(\theta) J_{|M|-1}(T\theta) + \beta_{LM}(\theta) J_{|M|}(T\theta)]\Big\} \qquad (111)$$

and

$$f_{LM}^{mult}(\theta) = (-)^{\frac{1}{2}(M-|M|)}\, i^{L+1}\, e^{i(2\sigma_T - \frac{1}{2}\theta_c)} \langle L_1 L_2 00 | L0 \rangle$$
$$\cdot \delta_1(L_1)\, \delta_1(L_2)\, \left(\frac{-ik}{2\pi^{\frac{1}{2}}}\right) [T/(4\pi^{\frac{1}{2}})] (\theta/\sin\theta)^{\frac{1}{2}}$$
$$\cdot \Big\{ [\phi_- H_-(\theta) + \phi_+ H_+(\theta)][\alpha_{LM}(\theta) J_{|M|}(T\theta) - \beta_{LM}(\theta) J_{|M|-1}(T\theta)]$$
$$+ i[\phi_- H_-(\theta) - \phi_+ H_+(\theta)][\alpha_{LM}(\theta) J_{|M|-1}(T\theta) + \beta_{LM}(\theta) J_{|M|}(T\theta)]\Big\}. \qquad (112)$$

2. <u>Discussion</u>

For the simplest SAM parametrization [10]

$$\eta_{\bar{\ell}} = g + i\mu \frac{dg}{d\ell} \quad , \quad g = [1 + e^{(T-\ell)/\Delta}]^{-1} \qquad (113)$$

eq. (107) gives

$$H_{\pm} = (1 + \mu\phi_{\pm})\, F(\pi\Delta\phi_{\pm}) \;, \qquad (114)$$

where

$$F(\pi\Delta\phi_{\pm}) = \frac{\pi\Delta\phi_{\pm}}{\sinh \pi\Delta\phi_{\pm}} \quad . \qquad (115)$$

The Clebsch-Gordan coefficients can be approximated by their asymptotic forms [23]

$$\langle \ell_0, L, -M\, M | \ell_0 + \lambda, 0 \rangle \approx d_{\lambda M}^{L}\left(\tfrac{\pi}{2}\right) \qquad (116)$$

where $d_{\lambda M}^{L}$ is an element of the rotation matrix. This approxi -

mation is valid under condition (73). The functions α_{LM} and β_{LM} now become universal functions of θ for a given L, M

$$\alpha_{LM}(\theta) \approx \sum_{\lambda=-L}^{L} i^{\lambda-L} d^{L}_{\lambda 0}(\tfrac{\pi}{2})\, d^{L}_{\lambda M}(\tfrac{\pi}{2}) \cos\tfrac{1}{2}\lambda\theta \tag{117}$$

$$\beta_{LM}(\theta) \approx \sum_{\lambda=-L}^{L} i^{\lambda-L} d^{L}_{\lambda 0}(\tfrac{\pi}{2})\, d^{L}_{\lambda M}(\tfrac{\pi}{2}) \sin\tfrac{1}{2}\lambda\theta \; . \tag{118}$$

From the symmetry relation [23]

$$d^{L}_{\lambda M}(\tfrac{\pi}{2}) = (-)^{L+M}\, d^{L}_{-\lambda,M}(\tfrac{\pi}{2}) \tag{119}$$

it follows

$$\begin{aligned} \alpha_{LM}(\theta) &= 0 \quad , \quad L+M \text{ odd} \\ \beta_{LM}(\theta) &= 0 \quad , \quad L+M \text{ even} \end{aligned} \tag{120}$$

2.1 Single excitation

From eqs. (36) and (105) and the property (120) one obtains

$$\begin{aligned} \frac{d\sigma}{d\Omega}(0\to L) = {} & [\delta_1(L)]^2 [T^2/(16\pi)](\theta/\sin\theta)\Big\{ (H_-^2+H_+^2) \\ & \cdot \sum_{M=-L}^{L} \left[\alpha^2_{LM}(\theta) + \beta^2_{LM}(\theta)\right]\left[J^2_{|M|}(T\theta) + J^2_{|M|-1}(T\theta)\right] \\ & + 2H_-H_+ \sum_{M=-L}^{L} \left[\alpha^2_{LM}(\theta) - \beta^2_{LM}(\theta)\right]\left[J^2_{|M|}(T\theta) - J^2_{|M|-1}(T\theta)\right]\Big\} . \end{aligned} \tag{121}$$

To discuss the features of this formula several special cases will now be considered.

a) Neutral particles

In the case of small Coulomb parameter the form factors $H_\pm$ become

$$H_\pm \approx (1+\mu\theta)\, F(\pi\Delta\theta) \; , \tag{122}$$

and eq. (121) reduces to

$$\frac{d\sigma}{d\Omega}(0\to L) = [\delta_1(L)]^2[T^2/(4\pi)](\theta/\sin\theta)[F(\pi\Delta\theta)]^2$$
$$\cdot\left\{\sum_{M=-L}^{L}\left[\alpha_{LM}^2(\theta)\,J_{|M|}^2(T\theta)+\beta_{LM}^2(\theta)\,J_{|M|-1}^2(T\theta)\right]\right.$$
$$\left.+(\mu\theta)^2\sum_{M=-L}^{L}\left[\alpha_{LM}^2(\theta)\,J_{|M|-1}^2(T\theta)+\beta_{LM}^2(\theta)\,J_{|M|}^2(T\theta)\right]\right\}. \qquad (123)$$

It is noticed that the rounding off in ℓ- space has the effect of multiplying the cross section by the square of the form factor F, which is unity for $\pi\Delta\theta=0$ and decreases monotonically with increasing $\pi\Delta\theta$. The Blair phase rule is obeyed. It follows (because of property (120)) that for even L only Bessel functions of even order contribute to the first term in curly brackets. Towards "large" angles $(T\theta\gg 1)$ they all oscillate in phase, so that the phase of the angular distribution is determined by the parity of L. The second term depends on $(\mu\theta)^2$ and therefore becomes important towards larger angles. It is as a whole out of phase with the leading term, and thus fills up the minima of the angular distribution. Comparison with the elastic scattering cross section for neutral particles [10]

$$\frac{d\sigma}{d\Omega_{el}} = \frac{T^2}{k^2}\,\frac{[F(\pi\Delta\theta)]^2}{\theta\sin\theta}\left[J_1^2(T\theta)+(\mu\theta)^2 J_0^2(T\theta)\right] \qquad (124)$$

shows that the oscillations in eq. (123) are in or out of phase with the elastic oscillations for odd and even L respectively.

For small angles such that $(\frac{1}{2}L\theta)^2 \ll 1$ it follows

$$\alpha_{LM}(\theta)\approx\alpha_{LM}(0)\quad,\quad\beta_{LM}(\theta)\approx\beta_{LM}(0)=0 \qquad (125)$$

where

$$\alpha_{LM}(0)=\tfrac{1}{2}\left[1+(-)^{L+M}\right]\frac{[(L-M)!\,(L+M)!]^{1/2}}{(L-M)!!\,(L+M)!!}\,. \qquad (126)$$

For these angles $(\mu\theta)^2 \ll 1$ and $F(\pi\Delta\theta) \approx 1$. Therefore eq. (123) reduces to the Blair Fraunhofer formula [2]

$$\frac{d\sigma}{d\Omega}(0\to L) = [\delta_1(L)]^2 \frac{T^2}{4\pi} \sum_{M=-L,-L+2,\dots} \frac{(L-M)!(L+M)!}{[(L-M)!!(L+M)!!]^2} J^2_{|M|}(T\theta). \quad (127)$$

b) Charged particles

In order to investigate the properties of the eq. (121) in the presence of Coulomb interaction, consider the region of angles $T\theta \gg |M|$. For these angles the following asymptotic result holds [32]

$$J_{|M|}(T\theta) \approx \sqrt{\frac{2}{\pi T\theta}} \cos(T\theta - |M|\pi/2 - \pi/4),$$

and eq. (121) becomes

$$\frac{d\sigma}{d\Omega}(0\to L) = [\delta_1(L)]^2 \frac{T}{8\pi^2 \sin\theta} \Big\{ (H_-^2 + H_+^2) \cdot \sum_{M=-L}^{L} [\alpha_{LM}^2(\theta) + \beta_{LM}^2(\theta)] + 2H_-H_+ \sin 2T\theta \cdot \sum_{M=-L}^{L} (-)^M [\alpha_{LM}^2(\theta) - \beta_{LM}^2(\theta)] \Big\}, \quad (128)$$

From the orthogonality relations [23]

$$\sum_{M=-L}^{L} d^L_{\lambda M}(\tfrac{\pi}{2})\, d^L_{\lambda' M}(\tfrac{\pi}{2}) = \delta_{\lambda,\lambda'}$$

$$\sum_{\lambda=-L}^{L} d^L_{\lambda 0}(\tfrac{\pi}{2})\, d^L_{\lambda 0}(\tfrac{\pi}{2}) = 1 \quad (129)$$

and the symmetry relation (119) it follows that

$$\frac{d\sigma}{d\Omega}(0\to L) = [\delta_1(L)]^2 \frac{T}{8\pi^2 \sin\theta} \left[H_-^2 + H_+^2 + (-)^L 2H_-H_+ \sin 2T\theta \right]. \quad (130)$$

This expression was first derived by Bassichis and Dar [14, 15] and independently by Hahne [16]. It consists of a smooth term $(H_-^2 + H_+^2)$ and an oscillatory term $(-)^L 2H_-H_+ \sin 2T\theta$.

The oscillations are bounded by the envelopes $(H_- - H_+)^2$ and $(H_- + H_+)^2$. The factor $(-)^L$ shows explicitly that the Blair phase rule is valid. Comparison of eq.(130) with the elastic scattering cross section in the asymptotic region [10]

$$\frac{d\sigma}{d\Omega}_{el.} = \frac{T^2}{2\pi k^2 \sin\theta}\left[\left(\frac{H_-}{\phi_-}\right)^2 + \left(\frac{H_+}{\phi_+}\right)^2 + 2\left(\frac{H_-}{\phi_-}\right)\left(\frac{H_+}{\phi_+}\right)\sin 2T\theta\right], \quad (131)$$

shows that the odd-L inelastic oscillations are in phase with the elastic oscillations. In contrast with the neutral case, the contributions from $Im\,\eta_\ell$ to both eqs. (130) and (131) are in phase with those from $Re\,\eta_\ell$ for angles $\theta < \theta'$ and out of phase only if $\theta > \theta'$, where

$$\theta' = \theta_c\left(\frac{2}{\mu\theta_c} + 1\right)^{\frac{1}{2}}. \quad (132)$$

If $H_+ \ll H_-$, eq. (130) reduces to

$$\frac{d\sigma}{d\Omega}(0\to L) \approx [\delta_1(L)]^2 \frac{T}{8\pi^2 \sin\theta}(1+\mu\phi_-)^2[F(\pi\Delta\phi_-)]^2 \quad (133)$$

which describes a smooth angular distribution, independent of L. It peaks at $\theta = \theta_c$ for $\mu = 0$. Equation (133) applies to the inelastic scattering of heavy ions by nuclei [14]. This result was first derived [33] (with $\mu = 0$ and $L = 0$) to describe the angular distributions of nuclear transfer reactions between heavy ions above the Coulomb barrier.

2.2 Double excitation

From eqs. (55), (111) and (120) it follows that

$$\frac{d\sigma}{d\Omega}(0\to L) = (2L+1)\left[C_2(L)\right]^2 \frac{T^2 k^2}{16\pi} \frac{\theta}{\sin\theta}$$

$$\cdot\Big\{ \left[\phi_-^2 H_-^2 + \phi_+^2 H_+^2\right] \sum_{M=-L}^{L} \left[\alpha_{LM}^2(\theta) + \beta_{LM}^2(\theta)\right]$$

$$\cdot\left[J_{|M|}^2(T\theta) + J_{|M|-1}^2(T\theta)\right] + 2\phi_-\phi_+ H_- H_+$$

$$\cdot\sum_{M=-L}^{L} \left[\alpha_{LM}^2(\theta) - \beta_{LM}^2(\theta)\right]\left[J_{|M|}^2(T\theta) - J_{|M|-1}^2(T\theta)\right]\Big\}. \tag{134}$$

In the neutral case ($\theta_c = 0$) eq. (134) reduces to

$$\frac{d\sigma}{d\Omega}(0\to L) = (2L+1)\left[C_2(L)\right]^2 \frac{k^2}{4} \frac{T^2}{4\pi} \frac{\theta}{\sin\theta}\left[\theta F(\pi\Delta\theta)\right]^2$$

$$\cdot\sum_{M=-L}^{L} \Big\{ \left[\alpha_{LM}^2(\theta) J_{|M|-1}^2(T\theta) + \beta_{LM}^2(\theta) J_{|M|}^2(T\theta)\right]$$

$$+ (\mu\theta)^2 \left[\alpha_{LM}^2(\theta) J_{|M|}^2(T\theta) + \beta_{LM}^2(\theta) J_{|M|-1}^2(T\theta)\right]\Big\}. \tag{135}$$

Comparison of eqs. (123) and (135) shows that the phase of the oscillations for a given multipole L is now reversed. The average slope of the double excitation angular distribution is flattened by a factor θ^2. The Blair Fraunhofer formula [12] is obtained at small angles:

$$\frac{d\sigma}{d\Omega}(0\to L) \approx (2L+1)\left[C_2(L)\right]^2 \frac{k^2}{4} \frac{T^2}{4\pi} \theta^2 \tag{136}$$

$$\cdot\sum_{M=-L,-L+2} \frac{(L-M)!\,(L+M)!}{\left[(L-M)!!\,(L+M)!!\right]^2} J_{|M|-1}^2(T\theta) .$$

V. ANALYSIS OF EXPERIMENTAL DATA

The closed formulae simplify the numerical computation of cross sections. This is particularly true for heavier projectiles and at high bombarding energies where direct summations over the large number of participating partial waves become increasingly time consuming, whereas the accuracy of the closed formulae improves.

The SAM parametrization (113) was applied with considerable success in a comprehensive analysis of elastic scattering data [22]. In order to obtain agreement over a wider range of angles, Springer and Harvey [34] introduced the following generalization of eq. (113):

$$\eta_{\bar{\ell}} = g + \lambda \frac{dg}{dt} + i\left(\mu_1 \frac{dg}{dt} + \mu_2 \frac{d^2 g}{dt^2}\right) , \tag{137}$$

where $T, \Delta, \lambda, \mu_1$ and μ_2 are free parameters. This form enabled them to obtain quite satisfactory simultaneous fits for their elastic and inelastic data of 50.9 MeV alpha particles scattered by ^{40}Ca. A summary is given below of the closed formulae which corresponds to the parametrization (137).

1. Elastic scattering

a) Non-identical particles

By applying the techniques of ref. [10] it follows that [35]

$$\frac{d\sigma}{d\Omega}_{el} = \left| f(\theta) \right|^2 \tag{138}$$

where

$$f(\theta) = \begin{cases} f_c(\theta) + f_n^{(-)}(\theta) & , \ \theta \leq \theta_c \\ f_n^{(+)}(\theta) & , \ \theta \geq \theta_c \end{cases} \tag{139}$$

with $f_c(\theta)$ the Coulomb scattering amplitude, and

$$f_n^{(\pm)}(\theta) = i\, e^{i\chi_c}\left(\frac{T}{2\pi k^2 \sin\theta}\right)^{1/2} \cdot \left\{ A^{\pm}(\theta)\, F(\pi\Delta\phi_-)\, e^{-i(T\theta - \pi/4)} - B(\theta)\, F(\pi\Delta\phi_+)\, e^{i(T\theta - \pi/4)} \right\}, \tag{140}$$

$$\begin{aligned} A^{\pm}(\theta) &= \frac{\pm n^{1/2}}{2\sin\frac{1}{2}\theta_c} G_{\pm}(\theta) - \mu_1 + i(\lambda + \mu_2 \phi_-) \\ B(\theta) &= 1/\phi_+ + \mu_1 - i(\lambda + \mu_2 \phi_+) \\ F(\pi\Delta\phi_{\pm}) &= \pi\Delta\phi_{\pm}/\sinh \pi\Delta\phi_{\pm}, \quad \phi_{\pm} = \theta_c \pm \theta \\ \theta_c &= 2\arctan(n/T), \quad \chi_c = T\theta_c - 2n\ln\sin\tfrac{1}{2}\theta_c + 2\sigma_0 . \end{aligned} \tag{141}$$

The functions $G_{\pm}$ are defined in ref. [10], and can be approximated by

$$G_{\pm} \approx -\frac{2\sin\frac{1}{2}\theta_c}{\pm n^{1/2}} \frac{1}{\phi_-} \tag{142}$$

if $|\phi_-| \gg \theta_c$.

b) Identical particles

The differential cross section is

$$\frac{d\sigma}{d\Omega}_{el.} = \left| f^{(s)}(\theta) \right|^2 \tag{143}$$

where

$$f^{(s)}(\theta) = f(\theta) + f(\pi - \theta) \tag{144}$$

and $f(\theta)$ is defined by eqs. (139-141).

2. Inelastic scattering

Introducing eq. (137) into eq. (107) gives

$$H_{\pm} = \left[\left(1+\mu_1 \phi_{\pm}\right) - i\phi_{\pm}\left(\lambda+\mu_2 \phi_{\pm}\right) \right] F(\pi \Delta \phi_{\pm}) \, . \tag{145}$$

Inspection of eqs. (105), (111) and (112) discloses that the scattering amplitudes $f^{(1)}_{LM}(\theta)$, $f^{(2)}_{LM}(\theta)$ and $f^{mut}_{LM}(\theta)$ can all be cast in the form

$$\begin{aligned} f_{LM}(\theta) = \; & \gamma_L \, (-)^{\frac{1}{2}(M-|M|)} \, i^{L+1} \, e^{i(2\sigma_T - \frac{1}{2}\theta_c)} \frac{T}{4\pi^{1/2}} \left(\frac{\theta}{\sin\theta}\right)^{1/2} \\ & \cdot \Big\{ (U_- + U_+) \left[\alpha_{LM}(\theta) \, J_{|M|}(T\theta) - \beta_{LM}(\theta) \, J_{|M|-1}(T\theta) \right] \\ & + i (U_- - U_+) \left[\alpha_{LM}(\theta) \, J_{|M|-1}(T\theta) + \beta_{LM}(\theta) \, J_{|M|}(T\theta) \right] \Big\} \end{aligned} \tag{146}$$

where the normalization factor γ_L and the functions $U_{\pm}$ depend on the type of excitation. The different cross sections are summarised in Table 2 in terms of $f_{LM}(\theta)$ defined above, with γ_L and $U_{\pm}$ specified for each process. The definition (106) can be used for α_{LM} and β_{LM}, or the approximations (117) and (118) which have the advantage of being universal functions of θ for a given L,M. These approximate forms are used in the present analysis.

It should be recalled that the closed formulae can be applied subject to conditions (72), (73), (102) and (104). For identical particles the $f_{LM}(\pi-\theta)$ amplitude will therefore be inaccurately given by eq. (146) at small angles. Fortunately this amplitude is small in comparison with $f_{LM}(\theta)$ at these angles.

TABLE 2

Summary of formulae for inelastic scattering

	Type of excitation	Differential cross section	$U_{\pm}$	γ_L
Non-identical particles	single	$\frac{d\sigma}{d\Omega}(0\to L) = \sum_{M=-L}^{L} \lvert f_{LM}(\theta) \rvert^2$	$H_{\pm}$	$\delta_1(L)$
	double	$\frac{d\sigma}{d\Omega}(0\to L) = \sum_{M=-L}^{L} \lvert f_{LM}(\theta) \rvert^2$	$i\phi_{\pm} H_{\pm}$	$-\frac{k}{2}(2L+1)^{\frac{1}{2}} C_2(L)$
	mutual	$\frac{d\sigma}{d\Omega}\begin{pmatrix} 0\to L_1 \\ 0\to L_2 \end{pmatrix} = \sum_{L=\lvert L_1-L_2\rvert}^{L_1+L_2} \sum_{M=-L}^{L} \lvert f_{LM}(\theta) \rvert^2$	$i\phi_{\pm} H_{\pm}$	$-\frac{k}{2}\pi^{-\frac{1}{2}} \langle L_1 L_2 00 \vert L0 \rangle \delta_1(L_1)\delta_1(L_2)$
Identical particles	single	$\frac{d\sigma}{d\Omega}(0\to L) = \sum_{M=-L}^{L} \lvert f_{LM}(\theta) + (-)^{L+M} f_{LM}(\pi-\theta) \rvert^2$	$H_{\pm}$	$\delta_1(L)$
	double	$\frac{d\sigma}{d\Omega}(0\to L) = \sum_{M=-L}^{L} \lvert f_{LM}(\theta) + (-)^{L+M} f_{LM}(\pi-\theta) \rvert^2$	$i\phi_{\pm} H_{\pm}$	$-\frac{k}{2}(2L+1)^{1/2} C_2(L)$
	mutual	$\frac{d\sigma}{d\Omega}\begin{pmatrix} 0\to L_1 \\ 0\to L_1 \end{pmatrix} = \sum_{L=\lvert L_1-L_2\rvert}^{L_1+L_2} \sum_{M=-L}^{L} \lvert f_{LM}(\theta) + (-)^{L+M} f_{LM}(\pi-\theta) \rvert^2$	$i\phi_{\pm} H_{\pm}$	$-\frac{k}{2}\pi^{-\frac{1}{2}} \langle L_1 L_2 00 \vert L0 \rangle [\delta_1(L_1)]^2$

3. Results and discussion

A search routine has been written (based on eqs. (138-141)) which determines the best fit parameters of the model (137) from the elastic scattering data. These parameter values are then used to calculate the inelastic cross sections listed in Table 2. The deformation distance $\delta_1(L)$ is determined by normalization.

3.1 $^{40}Ca(\alpha,\alpha)^{40}Ca$

The data of Springer and Harvey [34] have been re-analysed [35]. The results are shown in figs. 4 and 5. Satisfactory agreement has been obtained for the majority of levels (except for the weakly excited states at 5.90 MeV, 8.11 MeV and 8.59 MeV). The phases of the diffraction oscillations and the average slope of the cross sections are well reproduced. The agreement is particularly good for the strongly enhanced levels at 3.73 MeV and 4.48 MeV. These results are interesting because it was anticipated in Section III that the cross section could be visibly overestimated at larger angles by the AB theory.

It is seen from figs. 4 and 5 that the multipolarity is determined mainly by the structure of the cross section at forward angles. Springer and Harvey considered an incoherent 2^+ admixture to either 1^- or 3^- excitation for the 6.94 MeV level, and decided on the 3^-, 2^+ combination. Comparison of these two cases in fig. 4 shows that the 1^-, 2^+ combination is strongly favoured at forward angles.

The deformation parameters for the present analysis are compared with the values obtained by Springer and Harvey in

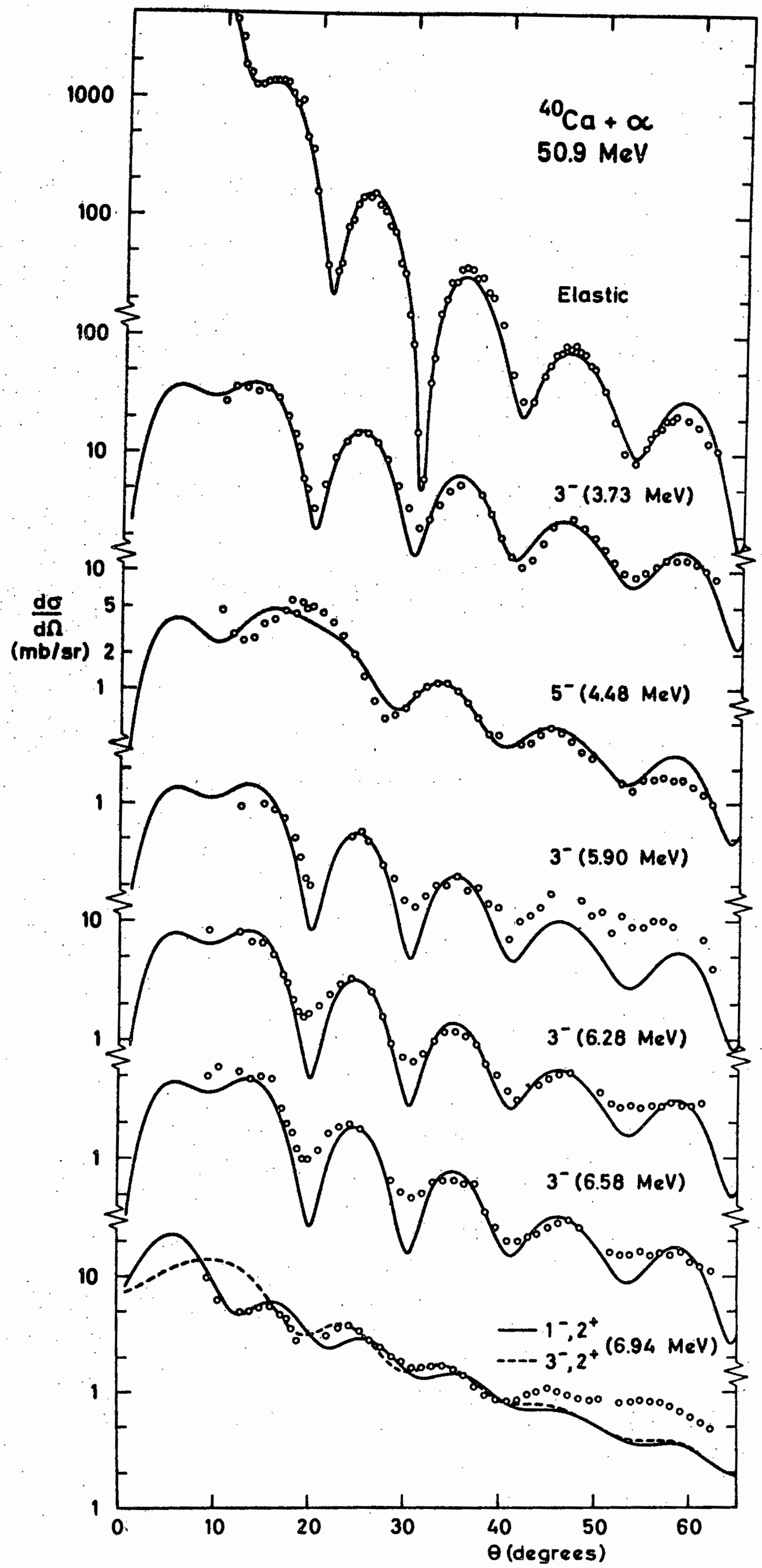

Fig. 4. Differential cross sections for elastic scattering and inelastic scattering from odd-parity levels, of 50.9 MeV alpha particles by ^{40}Ca.

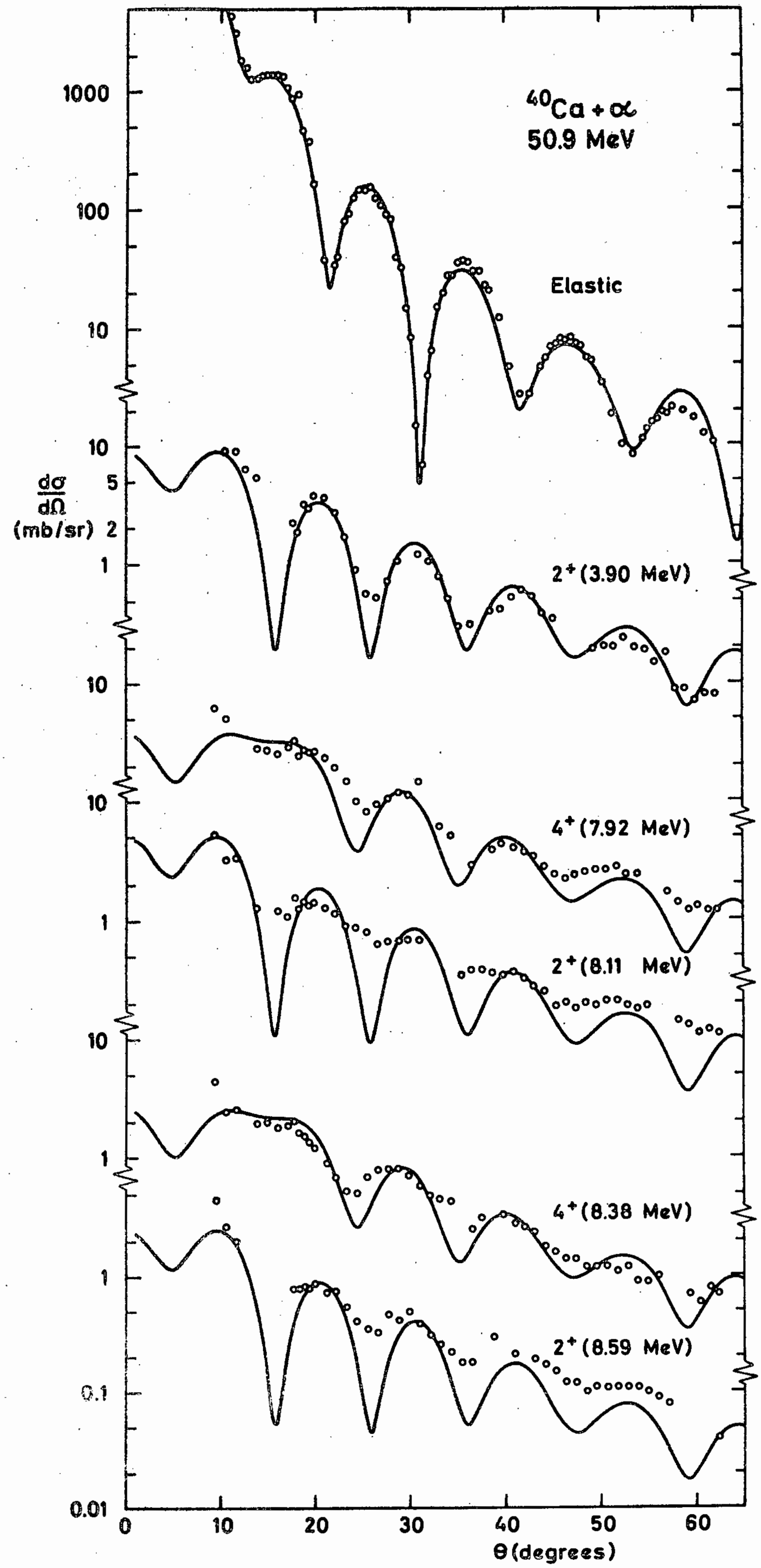

Fig. 5. Differential cross sections for elastic scattering and inelastic scattering from even-parity levels, of 50.9 MeV alpha particles by ^{40}Ca.

Table 4. The values agree closely. A DWBA analysis [36] of 31 MeV alphas scattered by ^{40}Ca gave $\delta_1 = 1{\cdot}08$ fm for the 3^- level. This value is about 20% bigger than the AB value [34] obtained at 50.9 MeV.

The accuracy of the closed formulae has been checked numerically. Agreement was found within a few percent for the lower multipoles.

3.2 $^{12}C(^{12}C,\,^{12}C)^{12}C$

The conditions under which the SAM formulae can be applied are particularly well satisfied for heavy ion scattering at laboratory energies of 10 MeV per nucleon or higher. Satisfactory agreement has been obtained for the data shown in fig. 6 [37]. The magnitude, phase and slope of the mutual excitation cross section are predicted correctly. The 4^+(14.05 MeV) level has been analysed by assuming a single or double excitation reaction mechanism. According to the rotational model, the reduced matrix element for double excitation is fixed through the relation [12]

$$C_2(L) = (2\pi^{1/2})^{-1} \langle L_1 L 0 0 | L_1 0 \rangle^2 \, [\delta_1(L_1)]^2, \tag{147}$$

where $\delta_1(L_1)$ refers to an intermediate state of multipolarity L_1. Although this model predicts the magnitude of the cross section correctly, the slope seems to be in favour of a single excitation mechanism.

3.3 $^{208}Pb(^{12}C,\,^{12}C)^{208}Pb$

The data shown in fig. 7 represent an example of the strong damping situation mentioned in Section IV. Because of the

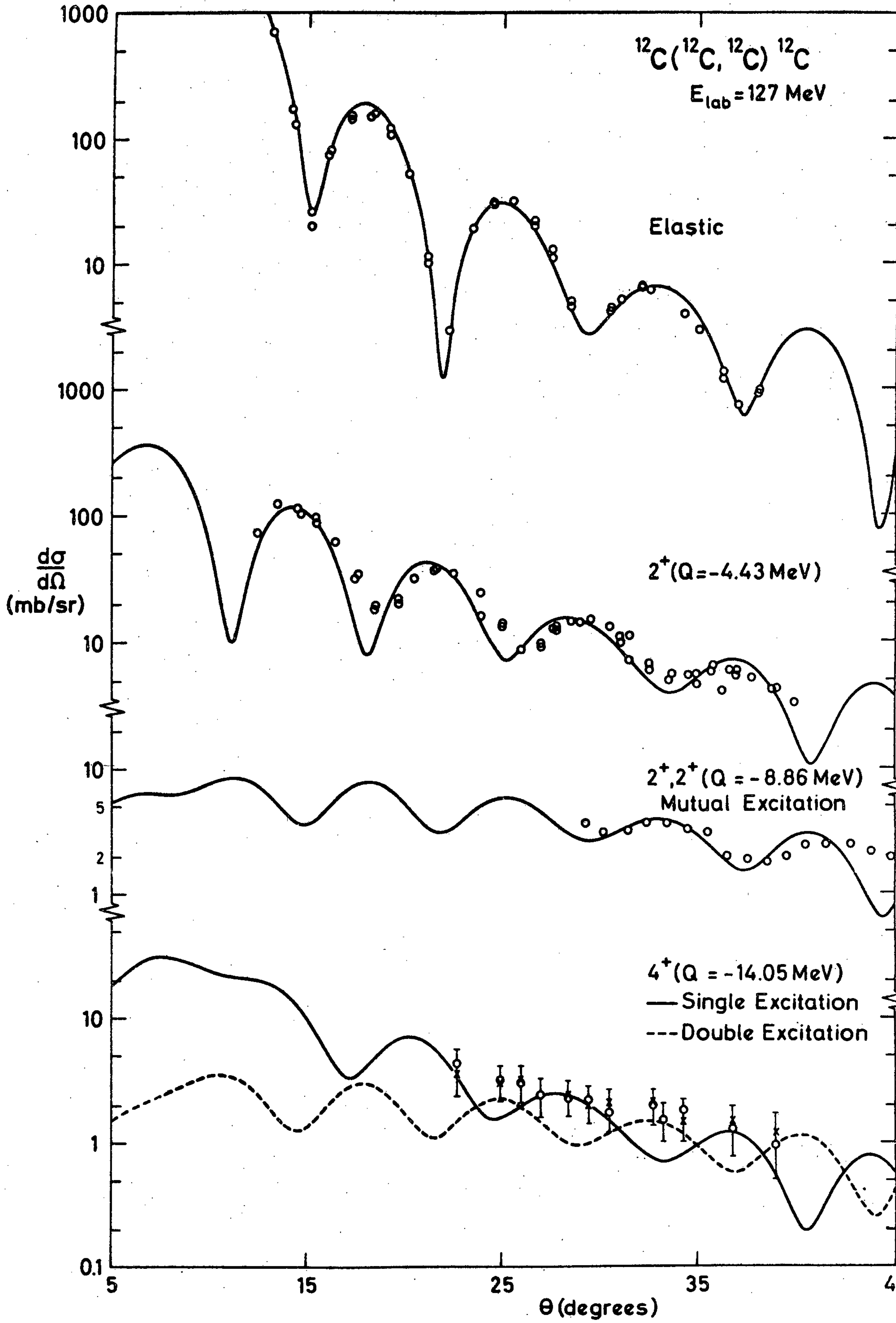

Fig.6. Differential cross sections for elastic and inelastic scattering of $^{12}C + ^{12}C$ at $E_{lab} =$ 127 MeV. The data are from refs. [38].

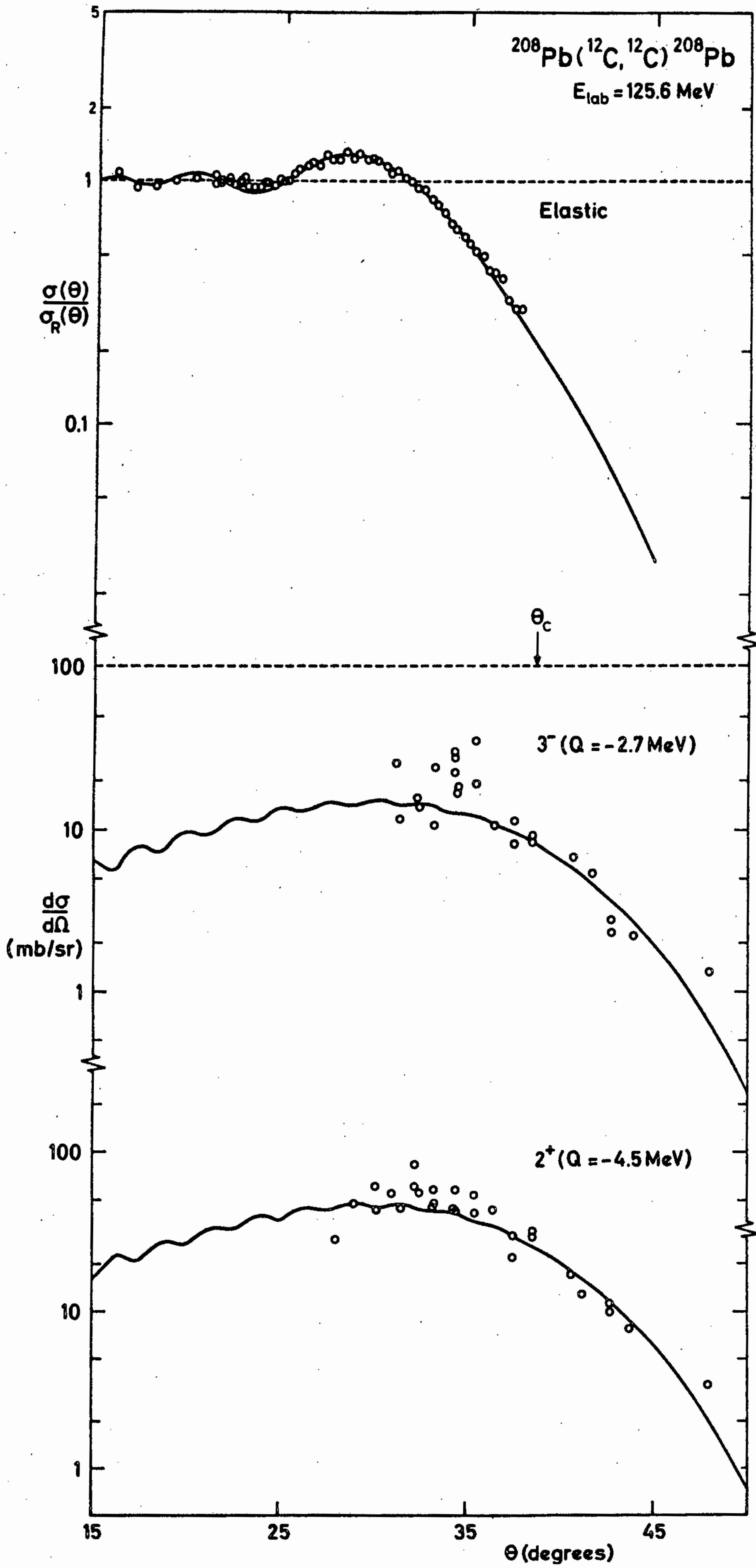

Fig.7. Differential cross sections for elastic and inelastic scattering of $^{12}C + ^{208}Pb$ at E_{lab} = 125.6 MeV. The data are from refs. [39, 40].

lack of diffraction structure it is not possible in this case to find unique values of the parameters Δ and μ_1 from the elastic cross section alone.

Since the AB and SAM assumptions are particularly well satisfied in this case (because of the large angular momenta involved), it was decided to search for Δ and μ_1 on the 2^+(4.5 MeV) inelastic angular distribution. The unambiguous value obtained for T in the elastic search was used. It was then possible to obtain a unique set of parameters Δ, μ_1 and $\delta_1(2^+)$. These parameter values were used to calculate all angular distributions, including the elastic cross section. Fig. 7 shows that the simultaneous fits obtained are satisfactory.

3.4 $^{12}C(^{16}O^{16}O)^{12}C$

The results of the analysis is shown in fig. 8 and is seen to be satisfactory. The mutual excitation cross section was calculated by using the deformation distances obtained for the 2^+(4.43 MeV) and 3^-(6.14 MeV) levels in carbon and oxygen respectively.

The SAM parameters for the elastic cross section are summarized in table 3. Their values are remarkably constant for all systems.

The deformation distances are listed in table 4. Identical values have been obtained for the 2^+(4.43 MeV) level in the $^{12}C + ^{12}C$ and $^{16}O + ^{12}C$ systems. It has been indicated that the major contribution to the 4.5 MeV level in the $^{12}C + ^{208}Pb$ system comes from the 4.43 MeV level in ^{12}C [40]. The value of $\delta_1(2^+) = 1.37$ fm agrees well with the value

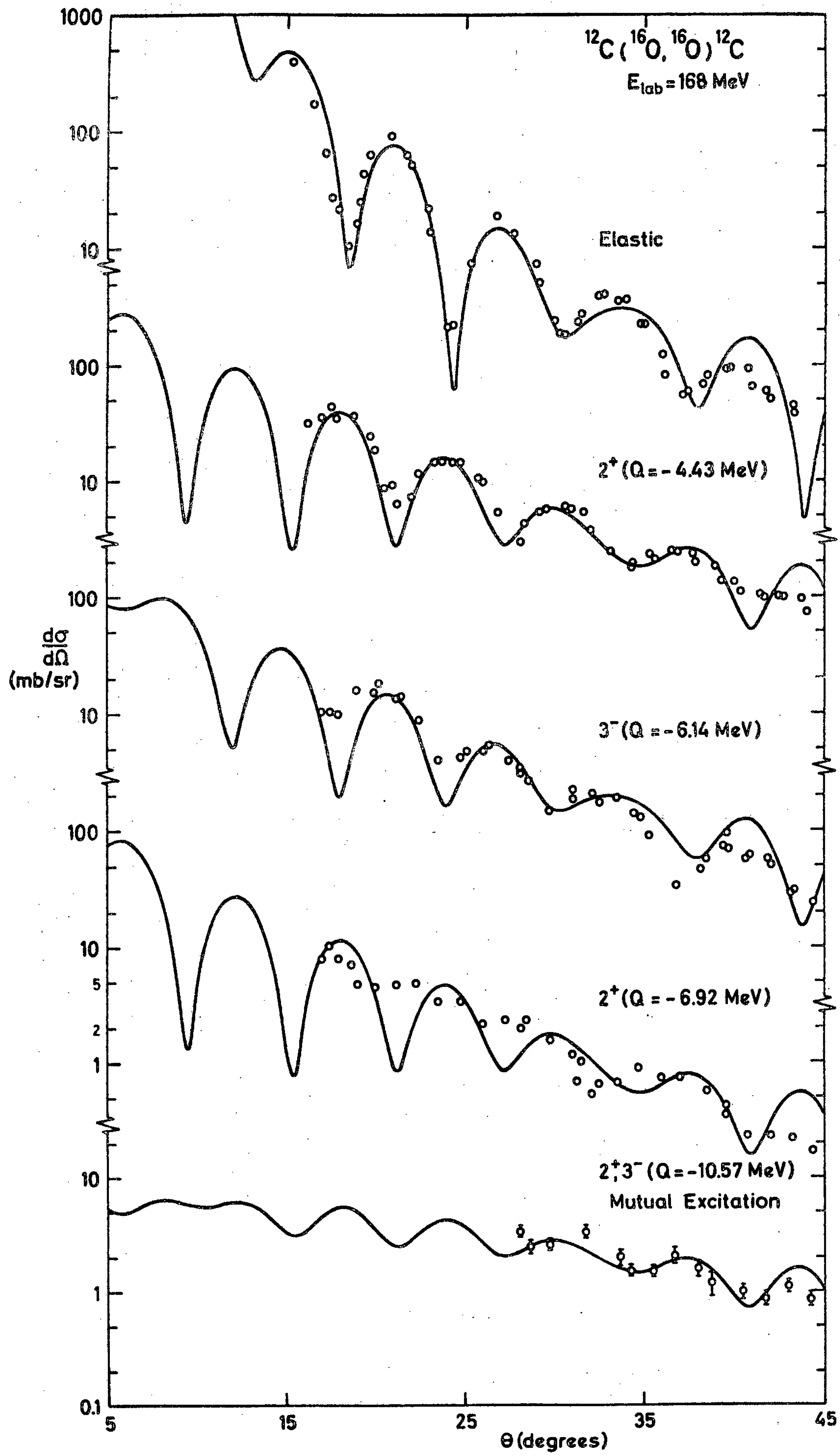

Fig. 8. Differential cross sections for elastic and inelastic scattering of $^{16}O + ^{12}C$ at $E_{lab} =$ 168 MeV. The data are from ref. [41].

TABLE 3

Strong absorption model parameters

System	E_{lab} (MeV)	Data Ref.	k $(fm)^{-1}$	n	T	Δ	$\frac{\lambda}{4\Delta}$	$\frac{\mu_1}{4\Delta}$	$\frac{\mu_2}{6\sqrt{3}\,\Delta^2}$	θ_c (deg)	r_o (fm)	d (fm)
$^{4}He + {}^{40}Ca$	50.9	34	2.84	1.77	17.6	1.13	-0.310	0.230	-0.20	11.5	1.37	0.40
$^{12}C + {}^{12}C$	127	38	4.27	1.74	26.6	1.46	-0.006	0.367	-0.07	7.5	1.45	0.34
$^{12}C + {}^{208}Pb$	125.6	39	8.03	24	68.3	2.68	0	0.330	0	38.6	1.46	0.32
$^{16}O + {}^{12}C$	168	41	4.83	2.33	30.5	1.68	-0.154	0.318	-0.11	8.7	1.41	0.35

TABLE 4

Deformation Distances

System	E_{lab} (MeV)	Data Ref.	E_x (MeV)	L^π	Excit.	$\delta_1(L)$ (fm)	δ_1 (fm) other auth.		
								method	Ref.
$^4He + ^{40}Ca$	50.9	34	3.73	$0^+,3^-$	S	0.83	0.85	AB	34
			3.90	$0^+,2^+$		0.33	0.34		
			4.48	$0^+,5^-$		0.35	0.35		
			5.90	$0^+,3^-$		0.16	0.18		
			6.28	$0^+,3^-$		0.38	0.40		
			6.58	$0^+,3^-$		0.29	0.31		
			6.94	$0^+,1^-+2^+$		0.38, 0.29	-		
				$0^+,3^-+2^+$		0.39, 0.30	0.36, 0.21		
			7.29	$0^+,4^+$		0.29	0.29		
			8.11	$0^+,2^+$		0.25	0.24		
			8.38	$0^+,4^+$		0.24	0.24		
			8.59	$0^+,2^+$		0.17	0.19		
$^{12}C + ^{12}C$	127	38	4.43	$0^+,2^+$	S	1.10	1.50	DWBA	43
			8.86	$2^+,2^+$	M	1.10, 1.10			
			14.05	$0^+,4^+$	S	0.44			
					D	1.10			
$^{12}C + ^{208}Pb$	125.6	40	2.7	$0^+,3^-$	S	0.77			
			4.5	$2^+,0^+$	S	1.37			
$^{16}O + ^{12}C$	168	41	4.43	$0^+,2^+$	S	1.10	1.69	DWBA	41
			6.14	$3^-,0^+$	S	0.82	1.35	DWBA	41
			6.92	$2^+,0^+$	S	0.61	0.90	DWBA	41
			10.57	$3^-,2^+$	M	0.82, 1.10			

found for the other two systems. For the 3^-(2.7 MeV) level in ^{208}Pb, a value of $\delta_l(3^-) = 0.77$ fm has been obtained. This value agrees closely with the result $\delta_l(3^-) = 0.75$ fm found by Alster [42] in an AB analysis of his 42 MeV alpha particle scattering data.

DWBA analyses were carried out for the $^{12}C + ^{12}C$ and $^{16}O + ^{12}C$ systems previously [43]. The DWBA values for δ_l are included in table 4. Comparison shows that the AB or SAM value for δ_l is consistently smaller than the corresponding DWBA value. Since in the DWBA analyses only the real part of the optical potential was deformed, the DWBA value for δ_l must be reduced by dividing with the factor $\sqrt{1+(W/V)^2}$, and the agreement is improved. For the 2^+(4.43 MeV) level the DWBA value becomes $\delta_l(2^+) = 1.3$ fm. The difference in this case still amounts to about 15% of the DWBA value.

Analyses of scattering data by means of the AB theory have been made by several authors [42, 44, 45]. The results obtained by Horen et al [44] are of particular interest. Deformation parameters were extracted for alpha particles scattered by ^{58}Ni and ^{62}Ni at incident energies of 33 MeV, 50 MeV, 85 MeV and 100 MeV. The deformation parameters obtained for the lowest quadrupole and octupole levels were found to be practically constant over the entire energy range in the case of both isotopes. It appears that the AB assumptions do not introduce an energy dependence into the deformation parameter.

SUMMARY

It has been shown that simple closed formulae can be obtained for the inelastic angular distributions from the Austern-Blair theory. All the well-known features of inelastic angular distributions of composite particles are explicitly described by these expressions. Furthermore, the numerical computation of differential cross sections are considerably simplified.

Analyses of representative sets of experimental data for alpha particles and heavy ions have shown that the closed expressions give satisfactory simultaneous fits of the angular distributions of elastic and inelastic scattering. Comparison with calculations in distorted wave Born approximation indicates that the deformation parameter obtained from the Austern-Blair theory is consistently smaller than the DWBA value.

APPENDIX I

From eq. (18) it can be shown that

$$\frac{d}{dr}\left(f_{e'} \frac{df_e}{dr} - f_e \frac{df_{e'}}{dr} \right) = \lambda(2\bar{e}+1)\frac{1}{r^2} f_e f_{e'} \tag{I.1}$$

where $\bar{e} = \frac{1}{2}(e+e')$, $\lambda = e - e'$. Constructing

$$\frac{df_{e'}}{dr}\frac{d^2 f_e}{dr^2} + \frac{df_e}{dr}\frac{d^2 f_{e'}}{dr^2} + \left[k^2 - \frac{2\mu}{k^2}(U+U_c)\right]\frac{d}{dr}(f_e f_{e'})$$
$$- \frac{1}{r^2}\left[e(e+1) f_e \frac{df_{e'}}{dr} + e'(e'+1) f_{e'} \frac{df_e}{dr} \right] = 0 \tag{I.2}$$

it can be shown that

$$\frac{d}{dr}\left(\frac{df_e}{dr}\frac{df_{e'}}{dr} \right) + \left[k^2 - \frac{\bar{e}(\bar{e}+1) + (\lambda/2)^2}{r^2} - \frac{2\mu}{k^2}(U+U_c) \right]$$
$$\cdot \frac{d}{dr}(f_e f_{e'}) - \frac{\lambda}{2}(2\bar{e}+1)\frac{1}{r^2}\left(f_e \frac{df_{e'}}{dr} - f_{e'}\frac{df_e}{dr} \right) = 0. \tag{I.3}$$

Integration gives

$$\Bigg| \frac{df_e}{dr}\frac{df_{e'}}{dr} + \Big\{ k^2 - \left[\bar{e}(\bar{e}+1) + \left(\frac{\lambda}{2}\right)^2 \right]\frac{1}{r^2}$$
$$- \frac{2\mu}{k^2}(U+U_c) \Big\} f_e f_{e'} - \frac{\lambda}{2}\frac{(2\bar{e}+1)}{r}\left(f_{e'}\frac{df_e}{dr} - f_e \frac{df_{e'}}{dr} \right) \Bigg|_0^\infty$$
$$+ \frac{2\mu}{k^2}\int_0^\infty dr \frac{\partial}{\partial r}(U+U_c) f_e f_{e'} + 2(\lambda^2-1)\,\bar{e}(\bar{e}+1)\int_0^\infty dr \frac{1}{r^3} f_e f_{e'} = 0, \tag{I.4}$$

Using the boundary conditions (19) and the property

$$\frac{df_e(k,0)}{dr} = 0 \quad , \quad e > 1 \tag{I.5}$$

gives

$$-\frac{1}{2} i E_{c.o.m.}\left(\eta_e e^{i\tau} - \eta_{e'} e^{-i\tau} \right) = \beta_{e',e}(k,k)$$
$$- \int_0^\infty dr \frac{\partial U_c}{\partial r} f_e f_{e'} - \frac{2E_{c.o.m.}}{k^2}(\lambda^2-1)\,\bar{e}(\bar{e}+1)\int_0^\infty dr \frac{1}{r^3} f_e f_{e'}$$

(I.6)

where $\tau = \sigma_\ell - \sigma_{\ell'} - (\lambda-1)\pi/2$. This expression can be transformed by means of eq. (15):

$$\beta_{\ell',\ell}(k,k) = -\tfrac{1}{2} i \, E_{c.o.m.} \Big\{ \Big[1 - \frac{2in}{\lambda(2\bar{\ell}+1)} \Big] \eta_\ell \, e^{i\tau} - \Big[1 + \frac{2in}{\lambda(2\bar{\ell}+1)} \Big] \eta_{\ell'} \, e^{-i\tau} \Big\} + \frac{2E_{c.o.m.}}{k^2} (\lambda^2-1) \bar{\ell}(\bar{\ell}+1) \cdot \int_0^\infty dr \, \frac{1}{r^3} f_\ell f_{\ell'} + \frac{2n}{k} E_{c.o.m.} \int_0^{R_c} dr \Big(\frac{1}{r^2} - \frac{r}{R_c^3} \Big) f_\ell f_{\ell'} , \tag{I.7}$$

Since the important contributions to $\beta_{\ell',\ell}$ come from the vicinity of $r \approx R_0 \gtrsim R_c$ the last term can be neglected for $\bar{\ell} \approx \ell_0$:

$$\beta_{\ell',\ell}(k,k) \approx -\tfrac{1}{2} i \, E_{c.o.m.} \Big\{ \Big[1 - \frac{2in}{\lambda(2\bar{\ell}+1)} \Big] \eta_\ell \, e^{i\tau} - \Big[1 + \frac{2in}{\lambda(2\bar{\ell}+1)} \Big] \eta_{\ell'} \, e^{-i\tau} \Big\} + \frac{2E_{c.o.m.}}{k^2} (\lambda^2-1) \bar{\ell}(\bar{\ell}+1) \int_0^\infty dr \, \frac{1}{r^3} f_\ell f_{\ell'} , \tag{I.8}$$

The right hand side still contains an unknown integral, but for $\lambda = \pm 1$ and $\ell \gg n$ the following interesting relation is obtained

$$\beta_{\ell-1,\ell} \approx -\tfrac{1}{2} i \, E_{c.o.m.} (\eta_\ell - \eta_{\ell-1}) . \tag{I.9}$$

APPENDIX II

Assuming that $l_0 + \frac{1}{2}$ is an integer and $\bar{l} > l_0 + \frac{1}{2}$, then it follows from eq. (21) that

$$\sigma_{\bar{l}} = \sigma_T + \sum_{s=T+1}^{\bar{l}} \arctan(n/s). \tag{II.1}$$

By defining an angle

$$\alpha_r = \arctan(n/T+r) - \arctan(n/T), \quad r = 1, 2, \ldots \tag{II.2}$$

and using assumption (72) it can be shown that

$$\alpha_r \approx \arctan\left[\frac{(n/T)(-r/T)}{1 + (n/T)^2(1 - r/T)}\right] \quad . \tag{II.3}$$

From eqs. (II.1) and (II.3) it follows that

$$\sigma_{\bar{l}} \approx \sigma_T - \frac{1}{4}\theta_c + \frac{1}{2}(t-T)\theta_c$$
$$+ \sum_{r=1}^{\bar{l}-T} \arctan\left[\frac{(n/T)(-r/T)}{1 + (n/T)^2(1 - r/T)}\right] . \tag{II.4}$$

Assumption (72) implies that the important contributions to the summation in eq. (37) come from the vicinity of $\bar{l} \approx T$. Hence it is possible to introduce the approximation

$$\sigma_{\bar{l}} \approx \sigma_T - \frac{1}{4}\theta_c + \frac{1}{2}(t-T)\theta_c \quad , \quad n \lesssim T \tag{II.5}$$

into eq. (37).

APPENDIX III

The spherical harmonics $Y_\ell^{-M}(\theta,0)$ are defined as

$$Y_\ell^{-M}(\theta,0) = (-)^{\frac{1}{2}(M-|M|)} \left[\frac{2\ell+1}{4\pi} \; \frac{(\ell-|M|)!}{(\ell+|M|)!} \right]^{1/2} P_\ell^{|M|}(\cos\theta) \;, \tag{III.1}$$

where $P_\ell^{|M|}(\cos\theta)$ is the associated Legendre function. For $M \geq 0$

$$P_\ell^{M}(\cos\theta) = \sin^M\theta \; \frac{d^M}{d(\cos\theta)^M} \; P_\ell(\cos\theta) \;, \tag{III.2}$$

The Legendre polynomials may be replaced by the first term of Szegö's expansion [46]

$$P_\ell(\cos\theta) \approx (\theta/\sin\theta)^{1/2} \, J_0[(\ell+\tfrac{1}{2})\theta] \;, \tag{III.3}$$

provided $(\theta^{-1} - \cot\theta) \ll 8(\ell+\frac{1}{2})$. From eqs. (III.2) and (III.3) it can be shown that

$$P_\ell^{M}(\cos\theta) = (\ell+\tfrac{1}{2})^M \, (\theta/\sin\theta)^{1/2} \Big\{ J_M[(\ell+\tfrac{1}{2})\theta]$$

$$- \frac{M^2}{2\ell+1}(\theta^{-1} - \cot\theta) \, J_{M-1}[(\ell+\tfrac{1}{2})\theta] + O[(\ell+\tfrac{1}{2})^{-2}] \Big\}. \tag{III.4}$$

Under the condition

$$\frac{M^2}{2\ell+1}\left(\theta^{-1} - \cot\theta\right) \ll 1 \tag{III.5}$$

eq. (III.4) can be approximated by the first term. Equation (III.1) becomes

$$Y_\ell^{-M}(\theta,0) \approx (-)^{\frac{1}{2}(M-|M|)} \left[\frac{2\ell+1}{4\pi} \; \frac{(\ell-|M|)!}{(\ell+|M|)!} \right]^{1/2} (\ell+\tfrac{1}{2})^{|M|} (\theta/\sin\theta)^{\frac{1}{2}} \tag{III.6}$$

$$\cdot \, J_{|M|}[(\ell+\tfrac{1}{2})\theta] \,.$$

APPENDIX IV

Equation (103) contains the series

$$S = \sum_{q=0}^{\infty} A_q$$

$$= \sum_{q=0}^{\infty} \frac{\Gamma(|M|-\frac{1}{2}+q)}{q!\,\Gamma(|M|-\frac{1}{2})} \left(-\Delta/T\right)^q (2\Delta\theta)^q \Big\{ J_{|M|-1+q}(T\theta)$$

$$\frac{\partial^q}{\partial[(\Delta\theta)^2]^q}\left[e^{-i\lambda\theta/2}\theta \frac{\pi\Delta\phi_+}{\sin \pi\Delta\phi_+} - e^{i\lambda\theta/2}\theta \frac{\pi\Delta\phi_-}{\sin \pi\Delta\phi_-}\right]$$

$$+ i\theta \frac{\partial^q}{\partial[(\Delta\theta)^2]^q}\left[e^{-i\lambda\theta/2} \frac{\pi\Delta\phi_+}{\sin \pi\Delta\phi_+} + e^{i\lambda\theta/2} \frac{\pi\Delta\phi_-}{\sin \pi\Delta\phi_-}\right] J_{|M|+q}(T\theta)\Big\}, \tag{IV.1}$$

Since this series converges rapidly it is sufficient to investigate the first two terms,

$$A_0 = J_{|M|-1}\left(e^{-i\lambda\theta/2}\theta \frac{\pi\Delta\phi_+}{\sin \pi\Delta\phi_+} - e^{i\lambda\theta/2}\theta \frac{\pi\Delta\phi_-}{\sin \pi\Delta\phi_-}\right)$$

$$+ i J_{|M|}\left(e^{-i\lambda\theta/2}\theta \frac{\pi\Delta\phi_+}{\sin \pi\Delta\phi_+} + e^{i\lambda\theta/2}\theta \frac{\pi\Delta\phi_-}{\sin \pi\Delta\phi_-}\right) \tag{IV.2}$$

and

$$A_1 = J_{|M|}\left[-\Delta(|M|-\tfrac{1}{2})/T\right]\Big\{ e^{-i\lambda\theta/2}\theta \frac{\pi\Delta\phi_+}{\sin \pi\Delta\phi_+}\Big[\frac{-i\lambda}{2\Delta} + \frac{1}{\Delta\theta}$$

$$+ \pi\left(\frac{1}{\pi\Delta\phi_+} - \coth \pi\Delta\phi_+\right)\Big] - e^{i\lambda\theta/2}\theta \frac{\pi\Delta\phi_-}{\sin \pi\Delta\phi_-}$$

$$\Big[\frac{i\lambda}{2\Delta} + \frac{1}{\Delta\theta} - \pi\left(\frac{1}{\pi\Delta\phi_-} - \coth \pi\Delta\phi_-\right)\Big] + i J_{|M|+1}$$

$$\cdot\left[-\Delta(|M|-\tfrac{1}{2})/T\right]\Big\{ e^{-i\lambda\theta/2}\theta \frac{\pi\Delta\phi_+}{\sin \pi\Delta\phi_+}\Big[-\frac{i\lambda}{2\Delta} + \pi\left(\frac{1}{\pi\Delta\phi_+} - \coth \pi\Delta\phi_+\right)\Big]$$

$$+ e^{i\lambda\theta/2}\theta \frac{\pi\Delta\phi_-}{\sin \pi\Delta\phi_-}\Big[\frac{i\lambda}{2\Delta} - \pi\left(\frac{1}{\pi\Delta\phi_-} - \coth \pi\Delta\phi_-\right)\Big]\Big\}, \tag{IV.3}$$

For small angles, such that $T\theta \ll 1$, the Bessel functions can be approximated by

$$J_\nu(z) \approx \frac{1}{\nu!}\left(z/2\right)^\nu . \tag{IV.4}$$

Comparison of eqs. (IV.2) and (IV.3) then shows that $A_1 \ll A_0$ if

$$\left| \left(|M| - \tfrac{1}{2}\right)\frac{\lambda}{2T} \right| \ll 1. \tag{IV.5}$$

At large angles where $T\theta \gg 1$, the Bessel functions can be approximated by

$$J_\nu(z) \approx \sqrt{\frac{2}{\pi z}}\, \cos\left(\pi z - \nu\pi/2 - \pi/4\right). \tag{IV.6}$$

Hence at angles where $J_{|M|}$ has maxima, $J_{|M|\pm 1}$ have minima. Comparison of the corresponding coefficients of $J_{|M|}$ in eqs. (IV.2) and (IV.3) shows that $A_1 \ll A_0$ under conditions (IV.5) and

$$\left| \frac{\pi\Delta\left(|M| - \frac{1}{2}\right)}{T}\left(\frac{1}{\pi\Delta\theta} \pm \frac{1}{\pi\Delta\phi_\pm} \mp \coth \pi\Delta\phi_\pm\right) \right| \ll 1. \tag{IV.7}$$

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